Topology of algebraic varieties

Wintersemester 2019/2020

0 Introduction

0.1 Overview

- motivating example of degenerating quadrics

1 Complex analysis

1.2 Topological spaces

- topological spaces
- continuous maps
- product topology
- quotient topology example: Zariski topology on $\mathbf{C}^n$
- gluing construction
- connected, Hausdorff, and quasi-compact spaces

1.3 Holomorphic functions

- Cauchy-Riemann equations and complex-differentiability [GH94, p. 2]
- for each open subset $U \subseteq \mathbf{C}^n$, a continuously real-differentiable function $f : U \to \mathbf{C}$ is *holomorphic* if it satisfies the following equivalent conditions [GH94, pp. 2-6]:
  (1) f is complex differentiable separately in each variable
  (2) f satisfies the Cauchy integral formula
  (3) f is analytic

1.4 Key theorems on analytic functions

- analytic implicit function theorem
- uniqueness of analytic continuation [Hör90, p. 5]
- maximum modulus principle [Hör90, p. 6]

2 Manifolds and vector bundles

2.5 Real manifolds

- real $\mathcal{C}^k$-manifolds [Voi07a, pp. 25, 39]
- $\mathcal{C}^k$-maps [Voi07a, p. 40]
- smooth partitions of unity for $X \subseteq \mathbf{R}^n$ [Lee13, pp. 43, 44]

2.6 Complex manifolds

- complex manifold [GH94, p. 14] or [Voi07a, p. 43]
- disks, polydisks, and $\mathbf{C}^n$ are pairwise nonisomorphic for $n > 1$
- Riemann surfaces [GH94, p. 15]
- $\mathbf{CP}^n$ [GH94, p. 15]
- complex tori [GH94, p. 16]
- smooth, projective algebraic varieties, the Jacobi criterion and the implicit function theorem

2.7 Tangent and cotangent bundles

- vector bundles [Voi07a, pp. 40, 41]
- tangent bundle and vector fields [Voi07a, pp. 41, 42]
- immersions and embeddings
- duals and exterior powers of vector bundles [Voi07a, p. 41]
- cotangent bundle and differential forms [Voi07a, p. 42]
- functoriality of tangent and cotangent bundles

2.8 Tangent bundles over complex manifolds

- holomorphic vector bundle [Voi07a, p. 43]
- holomorphic tangent bundle [GH94, pp. 16, 17] or [Voi07a, pp. 43-46]
- orientability of complex manifolds [GH94, p. 18]
- inverse and implicit function theorems [GH94, pp. 18, 19]

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3 Submersions and the Ehresmann fibration theorem

3.9 Constant-rank theorem

- submersions
- constant-rank theorem and local normal form
- fibers of submersions are submanifolds

3.10 Ehresmann fibration lemma

- local and global smooth flows
- integral curves of vector fields
- flows generated by vector fields
- locally trivial fibrations
- proper maps
- Ehresmann fibration lemma for $\mathcal{C}^\infty$-manifolds [Voi07a, pp. 221, 222]

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4 Real and holomorphic Morse theory

4.11 Key results

- statement of the Sard theorem [Hir94, pp. 68-72] or [Mil97, pp. 16-19]
- Hessian

- nondegenerate critical points [Voi07b, p. 42]
- ordinary double-point singularities [Voi07b, p. 43]
- real Morse lemma [Voi07b, pp. 20-23]
- holomorphic Morse lemma [Voi07b, p. 46]

4.12 Level sets

- level sets of Morse functions [Voi07b, pp. 23-27]

4.13 Vanishing cycles

- vanishing spheres and cycles [Voi07b, p. 47]
- topology of Lefschetz degenerations [Voi07b, pp. 48-51]

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5 Lefschetz pencils

5.14 Pencils and the Bertini theorem

- local holomorphic sections of line bundles and locally defined holomorphic functions
- vanishing loci of sections of line bundles and hypersurfaces
- pencil and base locus [Voi07b, p. 43]
- Bertini theorem [GH94, pp. 137, 138]

5.15 Lefschetz pencils

- the hyperplane bundle $\mathcal{O}_{\mathbf{P}^n}(1)$
- pencils of hyperplane sections [Voi07b, p. 43]
- Lefschetz pencil [Voi07b, p. 43]
- generic pencils of hyperplane sections are Lefschetz pencils [Lam81, pp. 19-22]

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6 Singular cohomology

6.16 Singular chains and cochains

- singular chains and homology
- singular cochains and cohomology
- relative homology and cohomology
- Eilenberg-Steenrod axioms
- Mayer-Vietoris sequence

6.17 Ingredients of Poincaré duality

- singular cochains with compact support and cohomology with compact support
- cup product
- cap product
- orientations revisited
- fundamental class

6.18 Poincaré duality

- Poincaré duality
- application: Gysin map

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7 Applications of holomorphic Morse theory

7.19 Blowing up the base locus

- blow-up [Voi07a, pp. 78-80]
- blow-up formula for cohomology [GH94, pp. 473, 474]
- blow-up of the base locus of a Lefschetz pencil [Voi07b, pp. 52-54]

7.20 Lefschetz hyperplane theorem

- Lefschetz hyperplane theorem [Voi07b, pp. 55-57]
- computation of cohomology of smooth hypersurface in $\mathbf{CP}^n$ except the middle degree
- description of the middle-degree cohomology [Voi07b, pp. 60-62]

7.21 Andreotti-Frankel theorem

- Andreotti-Frankel theorem [Voi07b, pp. 28-30]

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8 Monodromy and Picard-Lefschetz theory

8.22 Local Picard-Lefschetz formula

- monodromy representation
- local Picard-Lefschetz formula [Voi07b, pp. 79, 80, 82-84]

8.23 Global Picard-Lefschetz formula

- global Picard-Lefschetz formula [Voi07b, pp. 78, 79, 81, 82]

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9 Complex analytic spaces

9.24 Weierstaß preparation and division

- Hartogs extension theorem [GH94, p. 7]
- Weierstraß preparation theorem [GH94, pp. 7, 8]
- Riemann extension theorem [GH94, p. 9]
- Weierstraß division theorem [GH94, pp. 11, 12]

9.25 Chow's theorem

- Chow's theorem [Mum95, Ch. 4]

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