Topology of algebraic varieties

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Contents

8 Singular (co)homology

8.1 Homology

Let X be a topological space. Its *singular homology* with coefficients in the ring R is a sequence of R-modules

$$H_0(X, R), H_1(X, R), \dots$$

with many good properties. We will be mostly interested in the case $R = \mathbb{Q}$, but $R = \mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{F}_p$ are also standard possibilities. Cohomology is used to characterise topological spaces. If the cohomology groups are different, then so are the spaces. We sketch the construction and state the main properties. For details, see e.g. [Spa66].

The group $H_i(X, R)$ is defined as the i-th homology group of a complex $S_*(X, R)$ of R-modules.

Definition 8.1.1. (1) The *n-th standard simplex* Δ_n is the convex hull of the standard basis vectors $e_0, \dots, e_n$ of $\mathbb{R}^n$. For $i = 0, \dots, n$ let $\partial_i \Delta_n$ be the face spanned by omitting the vertex i. We identify it with Δ_{n-1}.

(2) Let X be a topological space A *singular n-simplex* on X is a continuous map $\sigma : \Delta_n \to X$. A *singular chain with coefficients in R* is a formal R-linear combination of singular n-simplices. Let $S_n(X, R)$ be the R-module of singular chains.

(3) We define $\partial : S_n(X, R) \to S_{n-1}(X, R)$ on the basis elements by

$$\partial\sigma = \sum_{i=1}^{n} (-1)^i \sigma|_{\partial_i \sigma}.$$

Elements in the kernel of ∂ are called *n-cycles*, elements in the image *n-boundaries.*

(4) We define *singular homology* as homology of the singular chain complex.

Exercise 8.1.2. Check that $S_*(X, R)$ is a complex of free R-modules.

Example 8.1.3. If $X = \star$ is a single point, then there is only one singular n-simplex, the constant map. Hence the chain complex is

$$\dots \to R \to \dots \to R \to 0$$

All ∂_i map the basis to a basis vector, hence ∂ alternatives betwen 0 or the identity, beginning with $0 : S_1(\star, R) \to S_1(\star, R)$. Hence

$$H_n(\star, R) = \begin{cases} R & n = 0, \\ 0 & n \neq 0. \end{cases}$$

Example 8.1.4. The 0-simplex is simply a point. Hence a singular 0-simplex is a point of X and $S_0(X, R)$ is the free module generated by the points of X.

The 1-simplex is an interval $[e_0, e_1]$. Then $\partial_0\Delta_1 = e_1$ (the end point), $\partial_1\Delta_1 = e_0$ (the starting point). A singular 1-simplex is a path. Its boundary is the formal difference of end point and starting point. A singular n-chain with coefficients in $\mathbb{Z}$ is a formal linear combination of paths. It is a cycle if every point appears as often as starting point as it appears as end point (counted with multiplicity). E.g. every closed path is a cycle. Let $\sigma : \Delta_2 \to X$ be a singular 2-simplex. Let x_0, x_1, x_2 be the images of the vertices of Δ_2. Then $\partial_0\sigma$ is a path from x_1 to x_2, $\partial_1\sigma$ is a path from x_0 to x_2 and $\partial_2\sigma$ is a path from x_0 to x_1. The sign in the definition of $\partial\sigma$ is chosen so that we go around the boundary of Δ_2.

Exercise 8.1.5. Let X be path connected, $x_0 \in X$. Then $H_0(X, \mathbb{Z}) = \mathbb{Z}$ and $H_1(X, \mathbb{Z})$ is the abelianized fundamental group.

Lemma 8.1.6. *Suppose X is contractible. Then $H_n(X, R)$ vanishes for $n \neq 0$ and $H_0(X, R) = R$.*

Proof. We explain the cases $n = 0, 1$. The general case is done by the same argument, but not as easy to draw. Let $x_0 \in X$ be fixed. By assumption, $\{x_0\} \subset X$ is a deformation retract.

All elements of $S_0(X, R)$ are cycles. Let $P \in X$ be a point. There is a path $\gamma : [0, 1] \to X$ with end points x_0 and P, hence $[P] - [x_0] = 0$ in $H_0(X, \mathbb{R})$. All generators of $S_0(X, R)$ define the same homology class. There are no additional relations, hence $H_0(X, R) \cong R[x_0]$.

Let $\sigma : \Delta_1 \to X$ be a singular 1-simplex. We can fill it in to a singular 2-simplex $\tilde{\sigma} : \Delta_2 \to X$ forming a a cone above σ with vertex x_0. We have

$$\partial\tilde{\sigma} = \sigma - [\partial_1\sigma, x_0] + [\partial_0\sigma, x_0]$$

(or some othere distribution of signs) where $[Q, x_0]$ denotes the path from Q to x_0 along the contraction. Now let $c = \sum_i a_i\sigma_i$ be a cycle. In $H_1(X, R)$ we may replace σ_i by $[\partial_1\sigma_i, x_0] - [\partial_0\sigma_i, x_0]$. We have

$$\sum_i a_i([\partial_1\sigma_i, x_0] - [\partial_0\sigma_i, x_0]) = 0$$

by the cycle condition for c. □

If there is not interesting topology, then homology vanishes. More generally:

Lemma 8.1.7. *Let $f, g : X \to Y$ be homotopy equivalent. Then $H_*(f) = H_*(g) : H_*(X, R) \to H_*(Y, R)$. In particular, homotopy equivalent spaces have the same homology.*

Proof. Similar to the special case of a contractible space. We use the homotopy to construct a chain homotopy between $S_*(f)$ and $S_*(g)$. □

The standard method for computing homology is by decomposing a topological space into smaller pieces. Let $X_1, X_2 \subset X$ be subspaces with $X_1 \cup X_2 = X$. We get a short exact sequence of complexes

$$0 \to S_*(X_1 \cap X_2, R) \xrightarrow{(i_1, i_2)} S_*(X_1, R) \oplus S_*(X_2, R) \xrightarrow{(a,b) \mapsto a-b} S_*(X_1, R) + S_*(X_2, R) \to 0$$

with

$$S_*(X_1, R) + S_*(X_2, R) \subset S_*(X, R).$$

Definition 8.1.8. The couple (X_1, X_2) is called *excisive* if the above map is an isomorphism on homology.

In this case, we get a long exact homology sequence

$$\cdots \to H_{n+1}(X, R) \to H_n(X_1 \cap X_2, R) \to H_n(X_1, R) \oplus H_n(X_2, R) \to H_n(X, R) \to \dots .$$

Theorem 8.1.9 ([Spa66, Ch. 4 Sec. 6 Thm 3])**.** *If $X = \mathrm{int}(X_1) \cup \mathrm{int}(X_2)$, then (X_1, X_2) is excisive.*

Proof. Omitted. The idea is to subdivide simplices until the pieces lie in X_1 or in X_2. This is possible by compactness of $\sigma(\Delta_n)$. □

Example 8.1.10. If $U_1 \cup U_2$ is an open cover of X, then it is excisive.

As an example we compute homology of spheres.

Proposition 8.1.11. *Let $n \geq 1$. Then*

$$H_i(S^n, R) = \begin{cases} R & i = 0, n \\ 0 & \textit{else.} \end{cases}$$

Proof. We argue by induction on n. The space S^0 consists of two disjoints points , hence $H_0(S^0, R) = \mathbb{R}^2$ (corresponding to the two points) and there is no higher homology. The 1-sphere can be coverd by two half circles intersecting in two intervals. Each of the pieces is contractible, so the long exact homology sequence gives

$$0 \to H_1(S_1, R) \to H_0(U_1 \cap U_2, R) \to H_0(U_1, R) \oplus H_0(U_2, R) \to H_0(S_1, R)$$

so we have to compute the kernel of the map $R^2 \to R^2$ given by $(a, b) \mapsto (a - b, a - b)$. It is isomorphic to R via $c \mapsto (c, c)$.

Now let $n > 1$. We cover S^n by two copies V_1, V_2 of B^n intersecing in a band around the equator. Hence $V_1 \cap V_2$ is homotopy equivalent to S^{n-1}. The Mayor-Vietoris sequence reads

$$0 \to H_n(S^n, R) \to H_{n-1}(S^{n-1}, R) \to 0 \oplus 0 \to \dots$$

□

8.2 Relative homology

There is another type of long exact sequence as well.

Definition 8.2.1. Let X be a topological space, $A \subset X$ a subspace. We put

$$S_*(X, A; R) = S_*(X, R)/S_*(A, R)$$

and call its homology $H_n(X, A; R)$ *relative singular homology* of (X, A).

Lemma 8.2.2. *There is a long exact sequence*

$$\dots H_n(A, R) \to H_n(X, R) \to H_n(X, A; R) \xrightarrow{\delta} H_{n-1}(A, R) \to \dots$$

Proof. By definition we have short exact sequence

$$0 \to S_*(A, R) \to S_*(X, R) \to S_*(X, A; R) \to 0.$$

Take its long exact sequence. $\square$

Example 8.2.3. Let $n \geq 1$. Then the long exact homology sequence gives an isomorphism $H_n(B^n, S^{n-1}, R) \to H_{n-1}(S^{n-1}, R)$. As both B^n and S^{n-1} are path connected, every class in $S_0(B^n, R)$ is in the image of $S_0(S^{n-1}, R)$ up to an element in the image of $S_1(B^n, R)$. Hence

$$H_i(B^n, S^{n-1}; R) = \begin{cases} R & i = n \\ 0 & i \neq n \end{cases}$$

Proposition 8.2.4 (Excision)**.** *Let (X, A) be a pair, $U \subset X$ open such $\bar{U}$ is contained in the interior of A. Then*

$$H_*(X - U, A - U; R) \cong H_*(X, A; R)$$

Proof. The pair $(X - U, A)$ is excisive by Theorem 8.1.9 because the

$$\mathrm{int}(X - U) \supset X - \bar{U} \supset X - \mathrm{int} A.$$

Hence by definition $S_*(X - U, R) + S_*(A, R) \to S_*(X, R)$ is an isomorphism on homology. Moreover,

$$S_*(X-U, R)+S_*(A, R)/S_*(A, R) \cong S_*(X-U, R)/S_*(X-U, R)\cap S_*(A, R) = S_*(X-U, A-U; R).$$

$\square$

There is a very usefull stronger version.

Theorem 8.2.5 ([Spa66, Ch. 4 Sect. 8 Thm 9])**.** *Let X be a compact Hausdorff space, $A \subset X$ a closed subset which is a deformation retract of one of its closed neighbourhoods in X. Let Y be Haussdorff, $B \subset Y$ closed. Let $f : (X, A) \to (Y, B)$ be continuous such that it induces a homeomorphism $X - A \to Y - B$. Then f induces an isomorphism on homology.*

Proof. The argument uses the deformation and clever excisison for (X, A) and (Y, B). $\square$

Example 8.2.6. Let X be a disjoint union of copies of B^n and A its boundary, hence a disjoint union of S^{n-1}. Then the assumptions on (X, A) in the theorem are satisfied.

Corollary 8.2.7. *Let X be a finite cell complex, X_n its n-skeleton. Suppose that X_n is obtained from X_{n-1} by gluing in m copies of B^n. Then*

$$H_i(X_n, X_{n-1}, R) = \begin{cases} R^m & i = n \\ 0 & i \neq n \end{cases}$$

Proof. Apply our upgraded version of excision to $(\coprod B^n, \coprod S^{n-1}) \to (X_n, X_{n-1})$. Then use our computation of relative homology of balls. $\square$

Remark 8.2.8. For $n < n'$, the inclusion $X_n \subset X_{n'}$ gives a long exact sequence for homology. The system of these long exact sequences organize in what is called a *spectral sequence*. It allows us to compute homology of X. What we have done in the corollary is to compute its starting term.

The category of *topological pairs* or simply *pairs* has as objects pairs (X, A) of a topological space X and subspace $A \subset X$. Morphisms $f : (X, A) \to (Y, B)$ are continuous maps $f : X \to Y$ such that $f(A) \subset B$. By construction, all $H_n(-, R)$ are functors from the category of pairs to the category of R-modules and δ is a transformation of functors $(X, A) \mapsto H_n(X, A; R) \to H_{n-1}(A, R)$.

On good spaces, e.g., finite cell complexes, singular homology is uniquely determined by the *Eilenberg-Steenrod* axioms:

- *Homotopy axiom* If $f, g : (X, A) \to (Y, B)$ are homotopic, they induce the same map on relative homology.
- *Exactness axiom* The long exact sequence for relative homology holds.
- *Excision axiom* The excision property of Proposition 8.2.4 holds.
- *Dimension axiom* If X consists only one point, then there is a natural isomorphism

$$H_i(X) = \begin{cases} R & i = 0 \\ 0 & i \neq 0 \end{cases}$$

8.3 Cohomology

Cohomology is dual to homology.

Definition 8.3.1. Let X a topological space, R a ring. We call

$$S^*(X, R) = \mathrm{Hom}_{\mathbb{Z}}(S_*(X, \mathbb{Z}), R) = \mathrm{Hom}_R(S_*(X, R), R)$$

the *singular cochain complex.* An *singular cochain* is a map which assigns to every singular n-simplex an element of R. Elements of the kernel (image) of the coboundary map are calls *cocycles* (*coboundaries*).

The homology of the singular cochain complex is called *singular cohomology*

$$H^n(X, R) = H^n(S^*(X, R)).$$

Remark 8.3.2. If R is a field, then

$$H^n(X, R) = H_n(X, R)^\vee$$

because $-^\vee = \mathrm{Hom}_R(-, R)$ is exact. For $R = \mathbb{Z}$, we get short exact sequences

$$0 \to \mathrm{Ext}^1(H_{n-1}(X, \mathbb{Z}), \mathbb{Z}) \to H^n(X, \mathbb{Z}) \to \mathrm{Hom}(H_n(X, \mathbb{Z}), \mathbb{Z}) \to 0.$$

Example 8.3.3. Let $n \geq 1$. Then

$$H^i(S^n, R) = \begin{cases} R & i = 0, n \\ 0 & \text{else} \end{cases}$$

by the same argument as for homology or by passing to duals (works because $H_i(S^n, R)$ is free over R.

The definition extends to the relative case. The dual of $S_*(X, A; R) = S_*(X, R)/S_*(A, R)$ is the kernel of $S^*(X, R) \to S^*(A, R)$.

Definition 8.3.4. Let X be a topological space, $A \subset X$ a subspace. We define *relative cohomology* of (X, A) as

$$H^n(X, A; R) = H^n(S_*(X, A; R)).$$

We get a long exact sequence

$$\dots \to H^n(X,A;R) \to H^n(X,R) \to H^n(A,R) \to H^{n+1}(X,A;R) \to \dots$$

Properties like homotopy equivalence, Mayer-Vietoris and excision also hold for singular cohomology.

Cohomology has a ring structure. It is induces pairing

$$S^k(X,R) \times S^l(X,R) \to S^{k+l}(X,R)$$

defined as

$$(\varphi,\psi) \mapsto (\sigma \mapsto \varphi(\sigma|[e_0,\dots,e_k])\psi(\sigma|[e_k,\dots,e_{l+k}]))\,.$$

It is not hard to check ([Hat02, Lemma 3.6] how this formula behaves with respect to the boundary. From this we get that it induces an R-linear map

$$\cup : H^k(X,R) \times H^l(X,R) \to H^{k+l}(X,R).$$

We call it *cup-product.*

Our next aim is the formulation of Poincaré duality. This needs a few constructions. We are going to follow [Hat02].

Definition 8.3.5. Let $S^*_c(X,R)$ be the subcomplex of chains c such that there is a compact $K \subset X$ such that c vanishes on all chains in $X-K$. We call its cohomology

$$H^i_c(X,R) = H^i(S^*_c(X,R))$$

cohomology with compact support.

Remark 8.3.6.

- If X is compact, then by definition $H^i_c(X,R) = H^i(X,R)$.
- Alternatively, we may see cohomology with compact support as

  $$H^i_c(X,R) = \lim_K H^i(X, X-K;R).$$

  By excision, the right hand side only depends on a neighbourhood of K in X.

There is a pairing for $k \geq l$

$$\cap : S_k(X,R) \times S^l(X,R) \to S_{k-l}(X,R)$$

defined by

$$\sigma \cap \varphi = \varphi(\sigma|[e_0,\dots,e_l])\sigma|[e_l,\dots,e_k].$$

It is not hard to check ([Hat02, p. 240]) how it behaves under the differential. From this we get that it induces an R-linear map

$$H_k(X,R) \times H^l(X,R) \to H_{k-l}(X,R).$$

We call it *cap-product.*

8.4 Orientations

Consider an $\mathbb{R}$-vector space V of dimension d. An *orientation* on V is the choice of an equivalence class of bases where two bases are equivalent (define the same orientation) is the base change matrix has positive determinant. Altnatively: $\det(V) := \Lambda^d V$ is a one-dimensional vector space. An orientation is the choice of an $\mathbb{R}_{>0}$-equivalence classes of basis of $\det(V)$. This point of view makes clear that there are only two possible orientations of V.

Example 8.4.1. Let V be a $\mathbb{C}$-vector space of dimension 1 with basis $v \in V$. It gives rise to the $\mathbb{R}$-basis (v, iv). Consider the base change given by multiplication by $\alpha = a + ib \in \mathbb{C}^*$. Then the base change matrix over $\mathbb{R}$ is given by

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

It has determinant $a^2 + b^2$, so it is orientation preserving.

Exercise 8.4.2. Let V be $\mathbb{C}$-vector space. A complex basis $(e_1, \dots, e_d)$ gives rise to a real basis $(e_1, ie_1, \dots, e_d, ie_d)$. Show that any two $\mathbb{C}$-bases define the same real orientation.

Definition 8.4.3. Let X be a smooth manifold. An *orientation* is the choice of an orientation of TU for all charts $U \subset X$ such that the transition functions are orientation preserving. It is *orientable* if an orientation exists.

Exercise 8.4.4. $\mathbb{R}P^2$ is not orientable.

Lemma 8.4.5. *Let X be a complex manifold. Then X is orientable as a smooth manifold.*

Proof. Let $(\varphi_{ab})_{a,b}$ be the transition matrix from one holomorphic coordinate system to the other. The induced base change maps on tangent spaces are $\mathbb{C}$-linear, hence orientation preserving. □

There is also a homological characterisation of orientation. Let X be a smooth manifold of dimension d. We consider the *local homology* $H_d(X, X - \{x\}, R)$ of X at x. By excision, we may replace X by a ball in a coordinate chart, hence by homotopy equivalence

$$H_d(X, X - \{x\}; R) \cong H_d(B^d, B^d - \{0\}, R) \cong H_d(B^d, S^{d-1}; R) \cong R).$$

Definition 8.4.6. An *R-orientation* at x is the choice of a generator of $H_d(X, X - \{x\}, R)$. An orientation of X is a function $x \mapsto \mu_x$ assigning to each $x \in X$ an R-orientation satisfying the "local consistency" condition that each $x \in X$ has a neighborhood $U \subset X$ homeomorphic to a ball in $\mathbb{R}^n$ and containing a smaller open ball B of finite radius about x such that all the local orientations μ_y at points $y \in B$ are the images of one generator μ_B of $H_d(X, X - B; R) \cong H_d(U, U - B; R)$ under the natural maps $H_d(X, X - B; R) \to H_d(X, X - \{y\}, R)$.

Lemma 8.4.7. *Let X be an orientable manifold. Then the choice of an orientation induces an R-orientation for every R.*

Proof. It suffices to do the case $R = \mathbb{Z}$. Everything else is induced by base change $H_d(X, X - \{x\}, R) \cong H_d(X, X - \{d\}, \mathbb{Z}) \otimes R$. In the case $R = \mathbb{Z}$ the cohomology group has two possible generators because $\mathbb{Z}$ has the two generators ± 1 as $\mathbb{Z}$-module. The choice of an orientation defines compatible isomorphisms with $H_d(B^d, S^{d-1}, \mathbb{Z})$, hence an R-orientation. □

Theorem 8.4.8 ([Hat02, Thm 3.26])**.** *Let X be a connected compact orientable smooth manifold of dimension d. Then*

$$H_d(X, R) \to H_d(X, X - \{x\}, R)$$

is an isomorphism for all $x \in X$. Moreover, $H_n(X, R)$ vanishes for $n > d$.

Definition 8.4.9. A *fundamental class for X* is a class $[X] \in H_d(X, R)$ such that $[X]$ defines an R-orientation for all x.

Actually, the theorem follows from a version in the non-compact case.

Theorem 8.4.10 ([Hat02, Lem. 3.27])**.** *Let X be a connected orientable smooth manifold of dimension d. Let $K \subset X$ be compact. Then there is a unique class in $H_d(X, X - K; R)$ whose image in $H_d(X, X - \{x\}; R)$ agrees with the orientation for all $x \in K$.*

Moreover, $H_n(X, X - K; \mathbb{R}) = 0$ for $n > d$.

8.5 Poincaré duality

If X is a smooth compact orientable manifold of dimension d, then cap-product defines a map

$$[X]\cap : H^l(X,R) \to H_{d-l}(X,R)$$

for all $l \leq d$.

Theorem 8.5.1. *Let X be an connected orientable compact manifold, then the composition*

$$[X]\cap : H^l(X,R) \to H_{d-l}(X,R)$$

is an isomorphism.

Proof. See [Hat02, Thm 3.30]. □

If $R = F$ is a field, this takes the form

$$H^l(X,F) \cong H^{d-l}(X,F)^\vee.$$

This allows us to transport the covariant functoriality of homology to cohomolgy.

Definition 8.5.2. Let $Y \subset X$ be an inclusion of connected compact complex manifolds of complex codimension r. We call the composition

$$H^i(Y,\mathbb{Z}) \cong H_{2d_Y-i}(Y,\mathbb{Z}) \to H_{2d_Y-i}(X,\mathbb{Z}) \to H^{d_X-d_Y+i}(X,\mathbb{Z}) = H^{i+rr}(X,\mathbb{Z})$$

the *Gysin map*. The image of the standard generator of $H^0(Y,\mathbb{Z})$ in $H^{2r}(X,\mathbb{Z})$ is called *cycle class of Y*.

We also need the non-compact case.

Lemma 8.5.3. *Let $K \subset X$ be compact. Then the cap-product induces a pairing*

$$H_n(X, X-K;R) \times H^l(X,X-K;R) \to H_{n-l}(X,R).$$

They are compactible for varying K.

Proof. By definition elements of $H^l(X,X-K;R)$ are represented by cochains that vanish outside of K. Hence they pair to 0 with chains for $X-K$, so the map factors via $S_*(X,R)/S_*(X-K,R)$. □

Recall that an orientation of X induces orientations on all $H^d(X,X-K;R)$ for all compact K.

Theorem 8.5.4 (Poincaré duality in the non-compact case, [Hat02, Thm 3.35])**.** *Let X be an oriented connected smooth manifold of dimension d. Then the duality map*

$$H^n_c(X,R) = \lim_K H^n(X,X-K;R) \xrightarrow{[\eta_K]\cap} H_{d-n}(X,R)$$

is an isomorphism.

And finally, the version allowing singularities.

Theorem 8.5.5 (Poincaré duality in the singular case, [Hat02, Thm 3.44])**.** *Let X be an oriented connected smooth manifold of dimension d. $K \subset X$ compact and locally contractible Then there is an isomorphism*

$$H_n(X,X-K;R) \to H^{d-n}(K,R).$$

References

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[Spa66] Edwin H. Spanier. *Algebraic topology.* McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.