Topology of algebraic varieties

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6 Some elements of Morse theory

6.1 Morse Lemma in the smooth case

Morse theory is a method to understand the topological structure of a manifold by analyzing the fibres of a smooth map to $\mathbb{R}$.

Example 6.1.1. Let T be a torus, embedded into $\mathbb{R}^3$. Let $h : T \to \mathbb{R}$ be the height function. There are caps at the top and at the bottom where the fibre is a single circle. In between it is a union of two circles. There are four exceptional fibres: the top and bottom (a single point each) and the two point where the fibre looks like a figure 8. These are the point where the rank of dh changes from 1 to 0. There the rank is 1, we have a local fibration by the Ehresmann lemma. We are going to see that the simple shape of the fibres is not a coincidence but that can also be read of from dh.

Definition 6.1.2. Let X be a smooth manifold, $f : X \to \mathbb{R}$ a smooth function. A point $x \in X$ is called *critical point* if $df_x = 0$. A value $t \in \mathbb{R}$ is called *critical value* if there is a critical point x with $f(x) = t$.

In the non-critical points, we have $\mathrm{rk}(df_x) = 1$ and f is a submersion.

The set of critical points is small:

Theorem 6.1.3 (Sard)**.** *Let $f : X \to \mathbb{R}^n$ be a smooth map of smooth manifolds. Let $\Sigma \subset X$ be the set of points where df_x is not surjective. Then $f(\Sigma)$ has measure 0 with respect to the Lebesgue measure.*

Proof. See [Hir94, pp. 68-72], [Mil97, pp. 16-19], or Theorem 10.3.1. □

Example 6.1.4. In the torus case, we have four critical points. At the top, the function h is strictly decreasing, i.e., it has local maximum. At the bottom, it has a local minimum. The two other critical points are saddle points. The function increases in one direction and decreases in another. From calculus we know that we can decide this behaviour from considering the second derivatives or Hesse matrix.

Definition 6.1.5. Let X be a smooth manifold, $f : X \to \mathbb{R}$ smooth. Let $x \in X$ be a critical point. We define the bilinear form

$$\mathrm{Hess}_x(f) : T_xX \times T_xX \to \mathbb{R}$$

by $(\partial_{x_i}, \partial_{x_j}) \mapsto \frac{\partial^2 f}{\partial x_j \partial x_i}(x)$ for local coordinates $x_1, \dots, x_n$ near x.

Exercise 6.1.6. $\mathrm{Hess}_x(f)$ is independent from the choice of local coordinates because x is critical.

Note that $\mathrm{Hess}_x(f)$ is symmetric because f is C^2. Symmetric bilinear forms are classified. There is a basis of the vector space such that the representing matrix has the shape

$$\begin{pmatrix} -I_r & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here $r+s$ is the rank of the quadratic form and r is the *index*. (This deviates from the terminology used in some German Linear Algebra textbooks like Fischer, but seems to be the correct standard).

Definition 6.1.7. We say $x \in X$ is a *non-degenerate* critical point if $\mathrm{Hess}_x f$ is non-degenerate. The *Morse-index* $\mathrm{ind}_x f$ is the index of the quadratic form.

Example 6.1.8. Let $X = \mathbb{R}^n$, $f(x_1, \dots, x_n) = -\sum_{i=1}^r x_i^2 + \sum_{i=r+1}^{r+s} x_i^2$ has

$$df = (-2x_1, \dots, -2x_r, 2x_{r+1}, \dots, 2x_{r+s}, 0, \dots, 0).$$

The critical points are $x_1 = \cdots = x_{r+s} = 0$. They are non-degenerate if $r + s = n$, i.e., there is a single critical point. The Hesse matrix is already in diagonal form and has index r.

Proposition 6.1.9 (Morse lemma)**.** *Let $P \in X$ be a non-degenerate critical point of a function f. Then there is a neighbourhood U of P and coordinates $x_1, \dots, x_n$ on X such that*

$$f(x) = f(P) - \sum_{i=1}^{r} x_i^2 + \sum_{i=r+1}^{n} x_i^2$$

with $r = \mathrm{ind}_P f$.

Corollary 6.1.10. *If P is a non-degenerate critical point, then it is an isolated critical point, i.e., there is a neighbourhood U of P such that P is the only critical point of f in U.*

Proof. Compute df in the local coordinates of the Morse lemma. □

Proof of the Morse lemma. Without loss of generality $X \subset \mathbb{R}^n$, $P = 0$. Let $x_1, \dots, x_n$ be the standard coordinates on $\mathbb{R}^n$. There are smooth functions g_{ij} on a neighbourhood of 0 such that

$$f(x_1, \dots, x_n) = f(0) + \sum_{i=1}^{n} (\partial_i f)(0) x_i + \sum_{i,j} x_i x_j g_{ij}(x_1, \dots, x_n)$$

(beginning of the Taylor expansion). Without loss of generality $f(0) = 0$. By assumption 0 is critical, hence

$$f(x_1, \dots, x_n) = \sum_{i,j} x_i x_j g_{ij}(x_1, \dots, x_n).$$

Without loss of generality, the matrix (g_{ij}) is symmetric (replace g_{ij} by $1/2(g_{ij} + g_{ji})$). We think of it as a quadratic form that we want to diagonalise. We have

$$\frac{\partial f}{\partial x_k} = \sum_j x_j g_{kj} + \sum_i x_i g_{ik} + \sum_{ij} x_i x_j \frac{\partial g_{ij}}{\partial x_k} = 2 \sum_j x_j g_{kj} + \sum_{i,j} x_i x_j \frac{\partial g_{ij}}{\partial x_k}$$

and hence

$$\frac{\partial^2 f}{\partial x_l \partial x_k} = 2g_{kl} + \sum_j x_j \frac{\partial g_{kj}}{\partial x_l} + 2\sum_j x_j \frac{\partial g_{lj}}{\partial x_k} + \sum_{i,j} x_i x_j \frac{\partial^2 g_{ij}}{\partial x_l \partial x_k}$$

and evaluating at 0:

$$\frac{\partial^2 f}{\partial x_l \partial x_k} = 2g_{kl}(0).$$

They form a non-degenerate matrix. Let $A \in \mathrm{Gl}_n(\mathbb{R})$ be the matrix that transforms it into normal form,

$$A^t\,(g_{ij})\,A = \begin{pmatrix} -I_r & 0 \\ 0 & I_{n-r} \end{pmatrix}.$$

We make a linear change of coordinates $x' = A^{-1}x$. In the new coordinates

$$f(x') = x'^t A^t\,(g_{ij})\,Ax'$$

hence the new functions $g' = A^t g A$ satisfy

$$\left(g'_{ij}(0)\right) = \begin{pmatrix} -I_r & 0 \\ 0 & I_{n-r} \end{pmatrix}.$$

We replace $x_1, \dots, x_n$ by $x'_1, \dots, x'_n$ and drop the primes. The next step is a non-linear change of coordinates to remove the g_{ij} completely. We have to complete the squares.

We have $g_{11}(0) = \pm 1$. We consider the case $g_{11}(0) = 1$ first. Hence the function is positive in some neighbourhood of 0. We replace X by this neighbourhood. We put

$$y_1 = \sqrt{g_{11}}x_1 + \sqrt{g_{11}}^{-1} g_{12}x_2 + \dots + \sqrt{g_{11}}^{-1} g_{1n}x_n$$

and hence

$$y_1^2 = g_{11}x_1^2 + 2g_{12}x_1x_2 + \dots + 2g_{1n}g_{1n}x_1x_n + \dots$$

In the new coordinates $y_1, x_2, \dots, x_n$ we have

$$f = y_1^2 + \sum_{i,j>1} x_i x_j h_{ij}$$

with new functions h_{ij}. By induction, we find new coordinates $y_2, \dots, y_n$ such that f has the desired description.

If $g_{11}(0) = -1$, the function is negative on some neighbourhood of 0. We replace X by this neighbourhood. This time we use

$$y_1 = \sqrt{-g_{11}}x_1 + \sqrt{-g_{11}}^{-1} g_{12}x_2 + \dots \sqrt{-g_{11}} g_{1n}x_n$$

with

$$-y_1^2 = g_{11}x_1^2 + 2g_{12}x_1x_2 + \dots$$

The rest of the argument is as in the positive case. □

6.2 Holomorphic Morse lemma

We repeat the theory on the holomorphic setting.

Definition 6.2.1. Let X be a complex manifold, $f : X \to \mathbb{C}$ a holomorphic function. A point $x \in X$ is called *critical point* if $df_x = 0$. A value $t \in \mathbb{R}$ is called *critical value* if there is a critical point x with $f(x) = t$.

Remark 6.2.2. In the non-critical points, we have $\mathrm{rk}_{\mathbb{C}}(df_x) = 1$ and f is a submersion. If, in addition, f is proper, then we get a locally trivial fibration of smooth manifolds over the complement of the critical values.

Definition 6.2.3. Let X be a complex manifold, $f : X \to \mathbb{C}$ holomorphic. Let $x \in X$ be a critical point. We define the bilinear form

$$\mathrm{Hess}_x(f) : T_xX \times T_xX \to \mathbb{C}$$

by $(\partial_{z_i}, \partial_{z_j}) \mapsto \frac{\partial^2 f}{\partial z_j \partial z_i}(x)$ for local holomorphic coordinates $z_1, \ldots, z_n$ near x.

As in the real case, $\mathrm{Hess}_x(f)$ is independent from the choice of local coordinates because x is critical.

Note that again $\mathrm{Hess}_x(f)$ is symmetric. As a bilinear form over a complex vector space, it can be simplified: There is a basis of the vector space such that the representing matrix has the shape

$$\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}.$$

The symmetric bilinear form is non-degenerate if and only if $s = n$, or equivalently, the matrix of the bilinear form is invertible.

Definition 6.2.4. We say $x \in X$ is a *non-degenerate* critical point if $\mathrm{Hess}_x f$ is non-degenerate. In this case, we say that the fibre $X_{f(x)}$ has *an ordinary double point.*

The fibres with ordinary double points are singular (not manifolds), but the singularities are of the simplest possible type.

Example 6.2.5. Let $X = \mathbb{C}^n$, $f(z_1, \ldots, z_n) = \sum_{i=1}^s z_i^2$ has

$$df = (2z_1, \ldots, 2z_s, 0, \ldots, 0).$$

The critical points are $z_1 = \cdots = z_s = 0$. They are non-degenerate if $s = n$, i.e., there is a single critical point. The Hesse matrix is already in diagonal form. The fibre

$$X_0 = \left\{ z \in \mathbb{C}^n | \sum_{i=1}^s z_i^2 = 0 \right\}$$

has an ordinary double point in 0.

Proposition 6.2.6 (Morse lemma)**.** *Let $P \in X$ be a non-degenerate critical point of a holomorphic function f. Then there is a neighbourhood U of P and holomorphic coordinates $z_1, \ldots, z_n$ on X such that*

$$f(x) = f(P) - \sum_{i=1}^n z_i^2.$$

Proof. Same argument as in the real case. At the point where we take square roots, note that a function g with $g(0) \neq 0$ admits a square root on a small ball around 0. □

Corollary 6.2.7. *If P is a non-degenerate critical point, then it is an isolated critical point, i.e., there is a neighbourhood U of P such that P is the only critical point of f in U.*

Proof. Compute df in the local coordinates of the Morse lemma. □

6.3 Topology of level sets

Let $f : X \to \mathbb{R}$ be smooth. Our aim is to understand $X_{\leq M} = f^{-1}((-\infty, M])$ for $M \in \mathbb{R}$ or more generally $X_{[M_1,M_2]}$ for the preimage of $[M_1, M_2]$ for $M_1 \leq M_2$. We write $X_M = X_{[M,M]}$. They are called level sets.

Lemma 6.3.1. *If M is not a critical value of f, then $X_{\leq M}$ is a manifold with boundary, i.e., every point has a neighbourhood homeomorphic to an open subset of $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$. The boundary X_M is an $(n-1)$-dimensional manifold.*

Proof. The preimage of $(-\infty, M)$ is open in X, hence a manifold. Consider a point x_0 on X_M. By assumption, f is a submersive near x_0. By the constant rank theorem, this means that there is a coordinate chart at x_0 such that the map f becomes the projection to the first coordinate. This is the coordinate for the manifold with boundary $X_{\leq M}$. □

Definition 6.3.2. Let X be a smooth manifold. A function $f : X \to \mathbb{R}$ is called *Morse function* if it is exhaustive, i.e., for every $M \in \mathbb{R}$ the closed subset $X_{\leq M}$ is compact, and every critical point is non-degenerate.

In particular, f is proper. Hence if $[M_1, M_2]$ does not contain any critical values, then f defines a locally trivial fibration. Indeed, it is even trivial:

$$X_{[M_1,M_2]} \cong [M_1, M_2] \times K$$

for some compact fibre K.

The main result of Morse theory is the following:

Theorem 6.3.3. *Let X be a smooth manifold, $f : X \to \mathbb{R}$ a Morse function, $M \in \mathbb{R}$. Then $X_{(-\infty,M)}$ has the homotopy type of a finite cell complex.*

Actually, the theory also gives very precise information how the topological space looks like. But let us first review the notions from topology used in the theorem.

Definition 6.3.4. A *cell* of dimension n is a topological space B homeomorphic to a closed ball of radius 1.

A topological space X has the structure of *finite cell complex* if there is a sequence of closed subspaces $X_0 \subset X_1 \cdots \subset X_n = X$ such that

(1) X_0 is a finite set of isolated points (0-cells);

(2) for each i, there is finite set $\Delta_1, \dots, B_{k(i)}$ of i-cells, continuous maps $f_j : \partial B_j \to X_{i-1}$ and a homeomorphism

$$X_i = \left(X_{i-1} \amalg \coprod_{j=1}^{k(i)} B_j \right) / \sim$$

where $\sim$ is the equivalence relation generated by $x \sim f_j(x)$ for all $x \in \partial B_j$.

The subspace X_i is called the i-skeleton.

Example 6.3.5. The sphere S^n has a structure of finite cell complex with two cells. We put $X_0 = *$ a single point, $X_0 = X_1 = \cdots = X_{n-1}$ and glue in a single n-cell by mapping all its boundary to $*$.

Exercise 6.3.6. Let X, Y be finite cell complexes. Show that $X \times Y$ also has a structure of finite cell complex.

Definition 6.3.7. Let X, Y be topological spaces, $A \subset X$ closed. Let $f, g : X \to Y$ be continuous. A *homotopy from f to g relative to A* is a continuous map

$$H : X \times [0, 1] \to Y$$

such that

(1) $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$,
(2) $H(a,t) = H(a,0)$ for all $a \in A$, $t \in [0,1]$.

If there is a homotopy from f to g, we say that f and g are *homotopic*.

We often write $H_t(x)$ instead of $H(x,t)$.

Example 6.3.8. The fundamental group Y with base point y_0 is the set of homotopy classes of maps $\gamma : [0,1] \to Y$ with $\gamma(0) = \gamma(1) = y_0$ with respect to homotopies relative to $A = \{0,1\}$. Each intermediate path of the homotopy is itself a closed path from y_0 to y_0.

Definition 6.3.9. Let X, Y be topological space. A continuous map $f : X \to Y$ is a *homotopy equivalence* if there is a continuous map $g : Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$.

We will have a special case:

Definition 6.3.10. Let X be a topological space, $A \subset X$ a closed subset, $i : A \to X$ the inclusion. We say that A is a *deformation retraction* of X if there is a continuous map $r : X \to A$ with $r \circ i = r|_A = \mathrm{id}_A$ and a homotopy relative to A from $i \circ r$ to id_X.

Example 6.3.11. The origin is a deformation retract of $\mathbb{R}^n$ via the deformation $H(x,t) = tx$ ($t \in [0,1]$, $x \in \mathbb{R}^n$).

Example 6.3.12. S^1 is a deformation retract of $\mathbb{C}^*$ via the deformation

$$H(re^{i\varphi}, t) = (r + (1-r)t)e^{i\varphi}$$

Example 6.3.13. Let $f : X \to \mathbb{R}^n$ be a trivial fibration. Then $f^{-1}(0)$ is a deformation retract of X.

Proposition 6.3.14. *Let $f : X \to \mathbb{R}$ be a Morse function, λ a critical value. Let $\lambda < M$ such that f does not have a critical value in $(\lambda, M]$. Then for all $0 < \epsilon < M - \lambda$, the level set $X_{\leq \lambda+\epsilon}$ is a deformation retract of $X_{\leq M}$.*

Proof. By the Ehresmann theorem applied to the restriction of f to (an open neighbourhood of) $X_{[\lambda+\epsilon, M]}$ there is a diffeomorphism

$$X_{[\lambda+\epsilon,M]} \cong X_{\lambda+\epsilon} \times [\lambda+\epsilon, M]$$

The retraction $[\lambda+\epsilon, M]$ to $\{\lambda+\epsilon\}$ induces the deformation on the level set. It glues to the identity on $X_{\leq\lambda+\epsilon}$. □

In order to understand what happens at a critical point, we consider the case $X = \mathbb{R}^2$, $f : X \to \mathbb{R}$ given by $(x_1, x_1) \mapsto -x_1^2 + x_2^2$. The critical value is 0. Its preimage is

$$f^{-1}(0) = \{(x_1, x_2) | (x_2 - x_1)(x_2 + x_1) = 0\},$$

so the union of the diagonal lines. The preimage of $a \in \mathbb{R}$ is the hyperbola

$$f^{-1}(a) = \{(x_1, x_2) | (x_2 - x_1)(x_2 + x_1) = a\}.$$

For $a < 0$ is lies to the left and right of diagonal cross. For $a > 0$ is lies above and below it. Up to deformation, we get from $X_{\leq -\epsilon}$ to $X_{\leq\epsilon}$, by gluing in $[-\sqrt{\epsilon}, \sqrt{\epsilon}] \times \{0\}$.

More generally: Let $X = \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ the standard function of index r, i.e.,

$$f(x_1, \dots, x_n) = -\sum_{i=1}^{r} x_i^2 + \sum_{i=r+1}^{n} x_i^2.$$

Fix $\epsilon > 0$. The level set X_ϵ has two connected components, both diffeomorphic to $\mathbb{R}^{n-1}$. We want to compare $X_{\leq -\epsilon}$ and $X_{\leq \epsilon}$. The interval is replaced by the r-cell

$$B^r(\epsilon) := \left\{ (x_1, \dots, x_n) | \sum_{i=1}^r x_i^2 \leq \epsilon, x_{r+1} = \dots = x_n = 0 \right\}.$$

It is a ball of radius $\sqrt{\epsilon}$. It is obviously contained in $X_{\leq \epsilon}$. Its intersection with $B_{\leq -\epsilon}$ are the points

$$\left\{ (x_1, \dots, x_r, 0, \dots, 0|) | \sum_{i=1}^r x_i^2 \leq \epsilon, -\sum_{i=1}^r x_i^2 \leq -\epsilon \right\} = \partial B^r(\epsilon) =: S^r(\epsilon).$$

It is an r-sphere of radius $\sqrt{\epsilon}$.

Lemma 6.3.15. *With the notation above, $X_{\leq \epsilon}$ deformation retracts onto $X_{\leq -\epsilon} \cup B^r(\epsilon)$.*

Proof. We need to define a deformation from $X_{\leq \epsilon}$ to the subset. Its existence is obvious from looking at the picture. Both Voisin (see [Voi07, Proposition 1.12]) and Milnor (see [Mil63, p. 15-19], wonderful pictures) write them down explicit.

We use a third method, inspired by the differential geometric approach to Morse theory. Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be the negative gradient vector field of f, i.e.,

$$G(x_1, \dots, x_n) = -(\partial_1 f, \dots, \partial_n f) = (2x_1, \dots, 2x_r, -2x_{r+1}, \dots, -2x_n).$$

Let Φ be its flow. Going along the flow-lines reduces the value of f. They follow the direction of steepest decent. Note that 0 is a fixed point of the flow. Most flow lines tend to infinity for $t \to \pm\infty$. In particular, they connect a point in the level set X_ϵ with a point in $X_{-\epsilon}$ after a finite amount of time. Actually, this happens more quickly if we take x further away from the origin. There are two exceptional sets: on $\{(x_1, \dots, x_r, 0, \dots, 0)\}$ the point 0 is *repelling*, it is the limit for $t \to -\infty$. On $\{(0, \dots, 0, x_{r+1}, \dots, x_n)\}$ it is *attracting*, it is the limit for $t \to \infty$. Note that the repelling set (also called *unstable manifold*) is precisely, where we find out cell $B^r(\epsilon)$.

Hence we proceed as follows: We follow the flow Φ for a finite amount of time, until points on $X_{\leq \epsilon}$ with $|x| > R$ for some R have reached $X_{\leq -\epsilon}$. We then move points along the flow

$$(x, t) \mapsto (x_1, \dots, x_r, tx_{r+1}, \dots, tx_n).$$

In both cases we leave points fixed once they have reached $X_{\leq -\epsilon} \cup B^r(\epsilon)$. □

Theorem 6.3.16. *Let X be a smooth manifold, $f : X \to \mathbb{R}$ a Morse function, $\lambda \in \mathbb{R}$ a critical value. Let $x_1, \dots, x_m$ be the critical points for this value, $r_1, \dots, r_m$ their Morse indices. The there is $\epsilon > 0$ (small enough) such that there is a deformation retraction of $X_{\leq \lambda + \epsilon}$ to the union of $X_{\leq \lambda + \epsilon}$ with cells $B^{r_1}, \dots, B^{r_m}$ of dimensions $r_1, \dots, r_m$ glued to $X_{\leq \lambda - \epsilon}$ along their boundaries such these these boundaries remain disjoint.*

Proof. The idea is to use the negative gradient flow away from balls around the singularities. The values ϵ can be chosen small enough such that these exceptional balls are disjoint. Within the exceptional ball we use the deformation constructed in the lemma.

There is a problem with this approach: the gradient depends on the choice of coordinates. The invariant object is

$$df = \sum_i \partial_i f dx_i.$$

It is a differential form rather than a tangent vector. In the proof of the lemma we were making an explicit identification by identifying the basis ∂_i of $T\mathbb{R}^n$ with the basis dx_i of $T^*\mathbb{R}^n$. More abstractly, this means that we have chosen a scalar product. Globally, such an object is called a *Riemannian metric*. They exist by partition of unity and can be chosen to coincide with our chosen scalar product on our disjoint balls. Hence the proof is actually complete. □

6.4 Vanishing spheres

We can apply these considerations to a complex manifold X and a holomorphic function f. We write $g = \Re(f)$. If x is non-critical for f, then it is also non-critical for g because $\Re\colon \mathbb{C} \to \mathbb{R}$ is a submersion.

Lemma 6.4.1. *Let X be a complex manifold of dimension n, $f : X \to \mathbb{C}$ holomorphic. Let $x \in X$ be a non-degenerate critical point for f. Then x is also a non-degenerate critical point for $g = \Re(f)$ of Morse index n.*

Proof. We use the local coordinates of the holomorphic Morse lemma. Then

$$f(z_1, \ldots, z_n) = \sum_{j=1}^{n} z_j^2.$$

With $z_j = x_j + iy_j$ this implies

$$g(x_1, y_1, \ldots, x_n, y_n) = \sum_{j=1}^{n} (x_j^2 - y_j^2).$$

This is the normal form for the smooth Morse lemma and we read off the Morse index n. □

There are no holomorphic maps $f : X \to \mathbb{C}$ such that $\Re(f)$ is exhaustive. Instead we concentrate on the local behaviour.

Let $f : X \to V$ be proper holomorphic, $V \subset \mathbb{C}$ an open neighbourhood of 0. We put $\Delta \subset V$ a (small) open disk around 0 such that $\bar{\Delta} \subset V$, $\Delta^* = \Delta - \{0\}$ the pointed disk.

Theorem 6.4.2. *Let $X \to V \subset \mathbb{C}$ be holomorphic and proper, x_0 with $f(0)$ an ordinary double point and all other points non-critical. Let $t \in \Delta^*$. Then $X_\Delta = f^{-1}\Delta$ deformation retracts to $X_t \cup B_t^n$ where B_t^n is a (real) ball of dimension n glued to X_t along the* vanishing sphere *S_t^n.*

The idea is to show that

- X_Δ retracts to $X_{\Delta, g \leq \epsilon}$
- $X_{\Delta, g \leq -\epsilon}$ retracts to $X_{-\epsilon}$
- Morse theory tells us that we get from one to the other by gluing in an n-cell (because the Morse index is n)
- Any two fibres X_t are diffeomorphic.

We first concentrate on the situation near x_0. Hence we consider an open $U \subset X$ homeomorphic to a ball in $\mathbb{C}^n$ and local coordinates $z_1, \ldots, z_n$ on U centered at x_0 such that

$$f(z_1, \ldots, z_n) = \sum_{i=1}^{n} z_i^2.$$

For $t = se^{i\theta} \in \Delta^* = \Delta \smallsetminus \{0\}$, the fibre U_t of f contains the sphere

$$S_t^{n-1} = \left\{ z \in U | z_i = \sqrt{s} e^{\frac{1}{2}\theta} x_i, x_i \in \mathbb{R}, \sum_{i=1}^{n} x_i^2 = 1 \right\}.$$

The set $\{z \in U || f(x)| \leq s\}$ contains the ball

$$B_t^n = \left\{ z \in U | z_i = \sqrt{s} e^{\frac{1}{2}\theta} x_i, x_i \in \mathbb{R}, \sum_{i=1}^{n} x_i^2 \leq 1 \right\}.$$

If U is big compared to Δ, then these sets do not depend on U.

Definition 6.4.3. S_t^{n-1} is called *vanishing sphere* of the family $(U_t)_{t\in\Delta}$ and U_t^n is called *cone of the vanishing sphere*. Its boundary is S_t^{n-1}.

Remark 6.4.4. It t tends to 0, then the vanishing sphere tends to a single point, 0. Note that the sphere depends on the choice of coordinates. Once we have introduced homology, we will see that the homology class of the vanishing sphere, *the vanishing cycle*, is independent of this choice of coordinates (up to sign depending on the orientation).

Example 6.4.5. If $t = -\epsilon$, then $s = \epsilon$, $\theta = \pi$

$$B^n_{-\epsilon} = \left\{ z \in U | z_i = \sqrt{\epsilon} i x_i, x_i \in \mathbb{R}, \sum_{j=1}^n x_i^2 \leq 1 \right\}.$$

In the notation of the last section this is the ball $B^n(\epsilon)$ with respect to the function $g = \Re(f)$. Analogously, $S^{n-1}_{-\epsilon} = S^{n-1}(\epsilon)$.

Proof of Theorem 6.4.2. We consider the holomorphic proper map $f : X \to V$ and restrict to $X_\Delta = f^{-1}\Delta$. Without loss of generality, $t = -\epsilon$. We use an Ehresmann flow with respect to g. It exists as long a we avoid 0. It induces a deformation retract from $X_{\Delta,g\leq-\epsilon}$ to $X_{\Delta,g=-\epsilon}$ and from X_Δ to $X_{\Delta,g\leq\epsilon}$. We use the deformation from the smooth case from $X_{\Delta,g\leq\epsilon}$ to the gluing of $X_{\Delta,g=-\epsilon}$ to B^n_ϵ via the vanishing sphere.

The level set $X_{\Delta,g=-\epsilon}$ is the preimage under f of

$$\Delta_{-\epsilon} = \{z \in \Delta | \Re(z) = -\epsilon\} = \{z \in \Delta | z = -\epsilon + iy\}.$$

This is an interval of length depending on the size of Δ. Hence we can use the Ehresmann flow to define a deformation retract of $X_{\Delta,g=-\epsilon}$ to the fibre $X_{-\epsilon}$. □

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