Topology of algebraic varieties

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5 Topology of proper submersions

5.1 Constant-rank theorem

Much of this section follows the exposition of [Dun18, §4.3].

Proposition 5.1.1 (Local normal form for submersions)**.** *Let $f\colon X \to \mathbf{R}^n$ be a submersion of smooth manifolds, let $p \in X$, and let $m := \dim(X)$. There exists a smooth chart (U, φ) at p such that*

$$(f \circ \varphi^{-1})(t_1, \ldots, t_m) = (t_1, \ldots, t_n)$$

for each $(t_1, \ldots, t_m) \in \varphi(U)$.

Proof. If (U, φ) is a smooth chart at p, then we may replace X by U and f by $f \circ \varphi^{-1}$ at any point in the proof. In particular, we may assume without loss of generality that $X \subseteq \mathbf{R}^n$ is an open subspace.

Let $f_1, \ldots, f_n\colon X \to \mathbf{R}$ denote the component functions of f. By Remark 4.5.5, we have $m \geq n$. As f is a submersion, the Jacobian matrix of f has rank n at p. Replacing (X, f) by a suitable $(U, f \circ \varphi^{-1})$ once more, we may assume without loss of generality that the submatrix $(\partial f_k / \partial x_\ell(p))_{1 \leq k, \ell \leq n}$ is invertible.

Consider the smooth map $\varphi\colon X \to \mathbf{R}^m$ given by

$$\varphi(t) := (f_1(t), \ldots, f_n(t), t_{n+1}, \ldots, t_m)$$

for each $t = (t_1, \ldots, t_m) \in X$. The Jacobian matrix of φ at p with respect to the standard bases is given by

$$\begin{bmatrix} (\frac{\partial f_k}{\partial x_\ell}(p))_{1 \leq k, \ell \leq n} & * \\ 0 & I_{m-n}. \end{bmatrix}$$

In particular, $\mathrm{d}\varphi_p$ is an $\mathbf{R}$-linear isomorphism. By the inverse function theorem [add reference], φ is a diffeomorphism from an open neighborhood $p \in U \subseteq X$ to an open neighborhood $\varphi(p) \in \varphi(U) \subseteq \mathbf{R}^m$. For each $t \in \varphi(U)$, we have

$$(f \circ \varphi^{-1})(t) = (t_1, \ldots, t_n).$$

Indeed, we may write each $t \in \varphi(U)$ as $\varphi(u)$ for some $u \in U$, and we have

$$(t_1, \dots, t_n) = (\varphi_1(u), \dots, \varphi_n(u)) = (f_1(u), \dots, f_n(u)) = f(u) = f(\varphi^{-1}(u)),$$

where $\varphi_k\colon X \to \mathbf{R}$ is the kth component function of φ for $1 \leq k \leq n$. The pair (U, φ) is therefore the required smooth chart. □

Lemma 5.1.2. *Consider the following data and hypotheses:*

- $X \subseteq \mathbf{R}^m$ *and* $Y \subseteq \mathbf{R}^n$ *are open neighborhoods of the origins;*
- $f\colon X \to Y$ *is a smooth map of constant rank* $r \in \mathbf{Z}_{\geq 0}$*;*
- $f(0) = 0$*;*
- *the Jacobian matrix of* f *at* p *with respect to the standard bases is of the form*

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

with A *an invertible* $r \times r$ *matrix.*

There exist smooth charts (U, φ) *at* $p \in X$ *and* (V, ψ) *at* $q \in \mathbf{R}^n$ *such that*

$$(\psi \circ f \circ \varphi^{-1})(t_1, \dots, t_m) = (t_1, \dots, t_r, 0, \dots, 0)$$

for each $(t_1, \dots, t_m) \in \varphi(U) \subseteq \mathbf{R}^m$.

Proof. Recall that $r = \mathrm{rk}_x(f) \leq \min(m, n)$ by Remark 4.5.5. Let $f_1, \dots, f_n\colon X \to \mathbf{R}$ denote the component functions of f. By hypothesis, the matrix $A = (\partial f_k / \partial x_\ell(0))_{1 \leq k, \ell \leq r}$ is invertible. Consider the smooth map $\varphi\colon X \to \mathbf{R}^m$ given by

$$\varphi(t) \coloneqq (f_1(t), \dots, f_r(t), t_{r+1}, \dots, t_m)$$

for each $t = (t_1, \dots, t_m) \in X$. Note that $\varphi(0) = 0$ and that the Jacobian matrix of φ at 0 with respect to the standard bases is given by

$$\begin{bmatrix} (\frac{\partial f_k}{\partial x_\ell}(0))_{1 \leq k, \ell \leq r} & * \\ 0 & I_{m-r}. \end{bmatrix}$$

In particular, $\mathrm{d}\varphi_0$ is an $\mathbf{R}$-linear isomorphism. By the inverse function theorem [add reference], φ is a diffeomorphism on an open neighborhood $0 \in U \subseteq X$. For each $t \in \varphi(U)$, we have

$$(f \circ \varphi^{-1})(t) = (t_1, \dots, t_r, g_{r+1}(t), \dots, g_n(t)),$$

where $g_k \coloneqq f_k \circ \varphi^{-1}$ for each $r < k \leq n$. Indeed, we may write each $t \in \varphi(U)$ as $\varphi(u)$ for some $u \in U$, and we have

$$\begin{aligned}
&(t_1, \dots, t_r, f_{r+1}(\varphi^{-1}(t)), \dots, f_n(\varphi^{-1}(t))) \\
&\quad = (f_1(u), \dots, f_r(u), f_{r+1}(\varphi^{-1}(\varphi(u))), \dots, f_n(\varphi^{-1}(\varphi(u)))) \\
&\quad = (f_1(u), \dots, f_n(u)) \\
&\quad = (f_1(\varphi^{-1}(t)), \dots, f_n(\varphi^{-1}(t))) \\
&\quad = f(\varphi^{-1}(t)).
\end{aligned}$$

The Jacobian matrix of $f \circ \varphi^{-1}$ at each $t \in \varphi(U)$ is given by

$$\begin{bmatrix} I_r & 0 \\ * & (\frac{\partial g_k}{\partial x_\ell}(t))_{\substack{r < k \leq n \\ r < \ell \leq m}} \end{bmatrix}.$$

By [add reference], $\mathrm{d}(f \circ \varphi^{-1}) = \mathrm{d}f \circ \mathrm{d}\varphi^{-1}$. As φ^{-1} is a diffeomorphism on $\varphi(U)$, the rank of $f \circ \varphi^{-1}$ is equal to the rank of f, i.e., $f \circ \varphi^{-1}$ is of rank r on $\varphi(U)$. The form of our matrix representation of $\mathrm{d}(f \circ \varphi^{-1})$ therefore implies that $\partial g_k / \partial x_\ell = 0$ for each $r < k \leq n$ and each $r < \ell \leq m$. In other words, the functions g_k with $r < k \leq n$ are independent of $t_{r+1}, \dots, t_m$.

Let $\alpha\colon \mathbf{R}^n \to \mathbf{R}^m$ be the smooth map given by

$$\alpha(t_1, \dots, t_n) = (t_1, \dots, t_r, 0, \dots, 0).$$

For each $r < k \leq n$, we therefore have a well-defined smooth map $h_k\colon \alpha^{-1}(\varphi(U)) \to \mathbf{R}^n$ given by $h_k \coloneqq g_k \circ \alpha$. Consider the smooth map $\rho\colon \alpha^{-1}(\varphi(U)) \to \mathbf{R}^n$ given by

$$\rho(t) = (t_1, \dots, t_r, t_{r+1} + h_{r+1}(t), \dots, t_n + h_n(t)).$$

We have $\rho(0) = 0$ and the $\mathbf{R}$-linear map $\mathrm{d}\rho_0$ is represented by the matrix

$$\begin{bmatrix} I_r & 0 \\ * & I_{n-r} \end{bmatrix}.$$

Thus, ρ is a diffeomorphism on an open neighborhood $0 \in V \subseteq \alpha^{-1}(\varphi(U))$. Let $\psi\colon \rho(V) \to V$ denote the smooth map inverse to $\rho\colon V \to \rho(V)$.

To complete the proof, choose an open neighborhood $0 \in U' \subseteq \varphi(U)$ such that $(f \circ \varphi^{-1})(U') \subseteq V$. The smooth charts $(\varphi^{-1}(U'), \varphi)$ and (V, ψ) satisfy the required condition: for each $t \in U'$, we have

$$\begin{aligned} (\psi \circ f \circ \varphi^{-1})(t) &= (\rho|_V)^{-1}(t_1, \dots, t_r, g_{r+1}(t), \dots, g_n(t)) \\ &= (\rho|_V)^{-1}(t_1, \dots, t_r, h_{r+1}(t_1, \dots, t_r, 0, \dots, 0), \dots, h_n(t_1, \dots, t_r, 0, \dots, 0)) \\ &= (t_1, \dots, t_r, 0, \dots, 0), \end{aligned}$$

as required. ☐

Theorem 5.1.3 (Constant-rank)**.** *Consider the following data and hypotheses:*

- $f\colon X \to Y$ *is a smooth map of smooth manifolds of respective dimensions* m *and* n*;*
- $x \in X$ *and* $y \coloneqq f(x) \in Y$*; and*
- f *is of constant rank* r*.*

There exist smooth charts (U, φ) *at* x *and* (V, ψ) *at* y *such that*

$$(\psi \circ f \circ \varphi^{-1})(t_1, \dots, t_m) = (t_1, \dots, t_r, 0, \dots, 0)$$

for each $t = (t_1, \dots, t_m) \in \varphi(U) \subseteq \mathbf{R}^m$.

Proof. The assertion is local with respect to X and Y. Replacing X and Y by the codomains of smooth charts at x and y, respectively, we may assume without loss of generality that $X \subseteq \mathbf{R}^m$ and $Y \subseteq \mathbf{R}^n$ are open subspaces. Suitably modifying our charts, we may furthermore assume that $x = 0$ and $y = 0$.

By hypothesis, $\mathrm{d}f_0$ is of rank r. The matrix of $\mathrm{d}f_0$ with respect to the standard bases therefore contains an invertible $r \times r$ submatrix. Modifying our charts once more, we may assume that the Jacobian matrix satisfies the hypotheses of Lemma 5.1.2, and the claim follows. ☐

Example 5.1.4. Each submersion $f\colon X \to Y$ of smooth manifolds is of constant rank: indeed, $\mathrm{d}f_x$ is surjective for each x, hence of rank $\dim(Y)$. In this case, Theorem 5.1.3 tells us that, locally on X and Y, the map f looks like the standard projection $(x, y) \mapsto x\colon \mathbf{R}^m \times \mathbf{R}^r \to \mathbf{R}^m$.

Proposition 5.1.5. *Each submersion* $f\colon X \to Y$ *of smooth manifolds is an open map.*

Proof. By Example 5.1.4, there exist an open cover $\{V_\alpha\}_{\alpha\in A}$ of Y and, for each $\alpha \in A$, an open cover $\{U_{\alpha\beta}\}_{\beta\in B_\alpha}$ such that the map $f\colon U_{\alpha\beta} \to V_\beta$ is homeomorphic to a standard projection $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n$. Each such standard projection is open. As open subsets are stable under arbitrary unions, the claim follows. ☐

Example 5.1.6. Each immersion $f\colon X \to Y$ of smooth manifolds is of constant rank: indeed, $\mathrm{d}f_x$ is injective for each x, hence of rank $\dim(X)$. In this case, Theorem 5.1.3 tells us that, locally on X and Y, the map f looks like the standard inclusion $x \mapsto (x, 0)\colon \mathbf{R}^m \to \mathbf{R}^m \times \mathbf{R}^n$.

Definition 5.1.7. Let $f\colon X \to Y$ be a smooth map of smooth manifolds.

(1) Let $x \in X$. We say that x is a *regular point with respect to* f if $\mathrm{d}f_x$ is surjective. We say that x is a *critical point with respect to* f if it is not regular.

(2) Let $y \in Y$. We say that y is a *regular value of* f if each $x \in f^{-1}(y)$ is a regular point. In particular, if $f^{-1}(y) = \varnothing$, then y is a regular value. We say that y is a *critical value of* f if it is not a regular value.

Corollary 5.1.8. *Let $f\colon X \to Y$ be a smooth map of smooth manifolds of respective dimensions m and n, and let $y \in Y$ be a regular value such that $f^{-1}(y) \neq \varnothing$. The preimage $f^{-1}(y)$ admits a smooth structure of dimension $m - n$ such that the inclusion $f^{-1}(y) \hookrightarrow X$ is an embedding.*

5.2 Flows and vector fields

Much of this section follows the exposition of [Dun18, Chapter 7] and [Lee13, Chapter 9].

Flows

Definition 5.2.1. Let X be a smooth manifold. A *global flow on* X is a smooth map $\Phi\colon \mathbf{R}\times X \to X$ such that:

- for each $p \in X$, $\Phi(0, p) = p$; and
- for each $p \in X$, for each $s, t \in \mathbf{R}$, $\Phi(t, \Phi(s, p)) = \Phi(t + s, p)$.

Example 5.2.2. The function $L\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $L(s, t) := s + t$ is a flow on the smooth manifold $\mathbf{R}$.

Example 5.2.3. For each $t \in \mathbf{R}$, let $A(t)$ denote the 2×2 matrix

$$\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}.$$

The function $\Phi\colon \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$ given by $\Phi(t, v) = A(t)v$ is a global flow on $\mathbf{R}^2$.

Remark 5.2.4. Let X be a smooth manifold. The datum of a global flow on X is equivalent to that of a group morphism $\alpha\colon : \mathbf{R} \to \mathrm{Diff}(X)$, where $\mathrm{Diff}(X)$ is the group of diffeomorphisms $\varphi\colon X \to X$ with respect to composition.

Definition 5.2.5. Let X be a smooth manifold. A *flow domain of* X is a subset $D \subseteq \mathbf{R} \times X$ such that, for each $p \in X$, the subset $D_p := \{t \in \mathbf{R} \mid (t, p) \in D\} \subseteq \mathbf{R}$ is an open interval containing 0.

Definition 5.2.6. Let X be a smooth manifold. A *local flow on* X is a smooth map $\Phi\colon D \to X$ such that:

- $D \subseteq \mathbf{R} \times X$ is a flow domain;
- for each $p \in X$, $\Phi(0, p) = p$; and
- for each $p \in X$, for each $s \in D_p$, and each $t \in D_{\Phi(s,p)}$, if $t + s \in D_p$, then $\Phi(t, \Phi(s, p)) = \Phi(t + s, p)$.

The local flow Φ is *maximal* if there is no local flow $\Psi\colon E \to X$ such that $D \subsetneq E$ and $\Psi|_D = \Phi$.

Remark 5.2.7. In Example 5.2.17, we will see why the generality of local flows is needed.

Definition 5.2.8. Let X be a smooth manifold and let $\Phi\colon D \to X$ be a local flow. The *velocity field* $\frac{\partial \Phi}{\partial t}\colon X \to TX$ is the vector field given by

$$\frac{\partial \Phi}{\partial t}(p) = [t \mapsto \Phi(t,p)].$$

This is well defined since this tangent vector only depends on the values of the curve $\Phi(-,p)$ in a small neighborhood of p.

Exercise 5.2.9. Let X be a smooth manifold and let $\Phi\colon D \to X$ be a local flow. Show that the velocity field $\partial\Phi/\partial t\colon X \to TX$ is a smooth vector field.

Definition 5.2.10. Let X be a smooth manifold and let $\Phi\colon D \to X$ be a local flow. For each $p \in X$, the *flow line of* Φ *through* p is the path $\Phi_p\colon D_p \to X$ given by $\Phi_p(t) = \Phi(t,p)$.

Integral curves

Definition 5.2.11. Let $a, b \in \mathbf{R}$, let X be a smooth manifold, and let $\gamma\colon (a,b) \to X$ be a smooth map. The *velocity curve* $\partial\gamma/\partial t\colon (a,b) \to TX$ *of* γ is the map given by

$$\frac{\partial \gamma}{\partial t}\colon s \mapsto [t \mapsto \gamma(s+t)].$$

Exercise 5.2.12. With the notation and hypotheses of Definition 5.2.11, show that $\partial\gamma/\partial t$ is a smooth map.

Definition 5.2.13. Let X be a smooth manifold, let $\xi\colon X \to TX$ be a smooth vector field, and let $p \in X$. An *integral curve of* ξ *starting at* p is a smooth map $\gamma\colon J \to X$ such that:

- $J \subseteq \mathbf{R}$ is an open interval containing 0;
- $\gamma(0) = p$; and
- for each $s \in J$, $\frac{\partial \gamma}{\partial t}(s) = \xi(\gamma(s))$.

Example 5.2.14. Consider the vector field $\partial/\partial x$ on $\mathbf{R}^2$ and the point $p = (a,b) \in \mathbf{R}^2$. The map $\gamma\colon \mathbf{R} \to \mathbf{R}^2$ given by $\gamma(t) = (t+a, b)$ is an integral curve of $\partial/\partial x$ through p.

Example 5.2.15. Consider the vector field $\xi = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on $\mathbf{R}^2$. Finding an integral curve $\gamma\colon \mathbf{R} \to \mathbf{R}^2$ of ξ starting at $p = (a,b)$ amounts to solving an initial-value problem:

$$\gamma'(t) = \xi(\gamma(t)), \qquad \gamma(0) = (a,b).$$

Writing $\gamma(t) = (x(t), y(t))$, this system takes the form

$$x'(t)\frac{\partial}{\partial x}\Big|_{\gamma(t)} + y'(t)\frac{\partial}{\partial y}\Big|_{\gamma(t)} = x(t)\frac{\partial}{\partial x}\Big|_{\gamma(t)} + y(t)\frac{\partial}{\partial y}\Big|_{\gamma(t)}, \qquad x(0) = a, \qquad y(0) = b.$$

As $\partial/\partial x$ and $\partial/\partial y$ are linearly independent, this is equivalent to the system

$$x'(t) = x(t), \qquad y'(t) = y(t), \qquad x(0) = a, \qquad y(0) = b,$$

which admits the unique solution $\gamma(t) = (a\exp(t), b\exp(t))$.

Vector flows

Remark 5.2.16. Let X be a smooth manifold, let Φ be a local flow, and let $\partial\Phi/\partial t$ be the associated velocity field. An integral curve of $\partial\Phi/\partial t$ starting at $p \in X$ is a flow line Φ_p of Φ.

Unless X is quasi-compact, there may exist smooth vector fields on X that do not arise as velocity fields of global flows on X as the following example illustrates.

Example 5.2.17. Let $X = \mathbf{R}_{<0} \times \mathbf{R}$, let $\xi = \partial/\partial x$, and let $p = (a, b) \in X$. As in Example 5.2.14, the integral curve of ξ starting at p is given by $\gamma(t) = (t + a, b)$. However, $\gamma(t)$ is only defined for $t < -a$. If we had $\xi = \partial\Phi/\partial t$, then the integral curve of ξ starting at p would be the flow line Φ_p, and would therefore be defined at each time $t \in \mathbf{R}$.

Theorem 5.2.18 (Existence, uniqueness, smoothness)**.** *Let $U \subseteq \mathbf{R}^n$ be an open subset, let $\xi \colon U \to \mathbf{R}^n$ be a smooth map, and let $(t_0, x_0) \in \mathbf{R} \times U$. There exist $\varepsilon \in \mathbf{R}_{>0}$, an open neighborhood $x_0 \in U_0 \subseteq U$, and a smooth map $\Phi \colon (t_0 - \varepsilon, t_0 + \varepsilon) \times U_0 \to U$ such that, for each $x \in U_0$, $\gamma(t) \coloneqq \Phi(t, x)$ is the unique solution to*

$$\gamma'(t) = \xi(\gamma(t)), \qquad \gamma(t_0) = x$$

on $(t_0 - \varepsilon, t_0 + \varepsilon)$.

Proof. See [Lee13, Appendix D]. □

Lemma 5.2.19. *Let X be a smooth manifold, let $\xi \colon X \to TX$ be a smooth vector field, and let $\gamma_0, \gamma_1 \colon J \to X$ be integral curves to ξ defined on an open interval $J \subseteq \mathbf{R}$ containing 0. If there exists $t_0 \in J$ such that $\gamma_0(t_0) = \gamma_1(t_0)$, then $\gamma_0 = \gamma_1$.*

Proof. Let $S \coloneqq \{t \in J \mid \gamma_0(t) = \gamma_1(t)\}$ and let $t_0 \in S$. We claim that $S = J$. Let $\varphi \colon J \to X \times X$ be the continuous map induced by γ_0 and γ_1 by the universal property of the product (Proposition 1.2.1). Note that $S = \varphi^{-1}(\Delta_X)$, where $\Delta_X \subseteq X \times X$ is the diagonal. As X is a manifold, it is Hausdorff, so Δ_X is closed by Proposition 1.3.17, so S is closed in J.

For each $t_1 \in S$, γ_0 and γ_1 are solutions of the same initial-value problem in a coordinate chart around p. By Theorem 5.2.18, $\gamma_0 = \gamma_1$ on an open neighborhood of t_1, so S is also open in J.

As J is connected and S is a nonempty subset of J that is both open and closed, we have $S = J$. □

Theorem 5.2.20. *Let X be a smooth manifold and let $\xi \colon X \to TX$ be a smooth vector field, There exists a unique maximal local flow $\Phi \colon D \to X$ such that $\xi = \partial\Phi/\partial t$.*

Proof. Let $p \in X$. Let $A(p)$ denote the set of integral curves of ξ starting at p, and let $B(p)$ denote the set

$$B(p) \coloneqq \{J \subseteq \mathbf{R} \mid \exists(\gamma \colon J \to X) \in A(p)\}$$

be the set of domains of integral curves of ξ starting at p. By the existence assertion of Theorem 5.2.18, $A(p) \neq \varnothing$. Let

$$D_p \coloneqq \bigcup_{J \in B(p)} J \subseteq \mathbf{R}.$$

Define a map $\Phi_p \colon : D_p \to X$ by $\Phi_p(t) \coloneqq \gamma(t)$ for some $(\gamma \colon J \to X) \in A(p)$. By the uniqueness assertion of Theorem 5.2.18, Φ_p is well defined. Let $D(\xi) \coloneqq \{(t, p) \in \mathbf{R} \times X \mid t \in D_p\}$ and define a map $\Phi \colon D(\xi) \to X$ by $\Phi(t, p) \coloneqq \Phi_p(t)$.

It suffices to establish the following assertions:

(1) for each $p \in X$, $\Phi(0, p) = \Phi_p(0) = p$;

(2) for each $p \in X$ and each $s, t \in \mathbf{R}$, $\Phi(t)\Phi(s, p) = \Phi(t + s, p)$ whenever the left-hand side is defined;

(3) $D(\xi)$ is open in $\mathbf{R} \times X$;

(4) $\Phi\colon D(\xi) \to X$ is smooth.

Claim (1) is true by definition.

Consider Claim (2). Let $p \in X$, let $s \in D_p$, let $q := \Phi(s, p)$, and let $\gamma(t) := \Phi(t+s, p)$ whenever the latter is defined. By the chain rule, γ is an integral curve of ξ starting at q, so $\gamma = \Phi_q$ by Theorem 5.2.18. It follows that $\Phi(t+s, p) = \Phi(t, \Phi(s, p))$ when both sides are defined. Since Φ_p and Φ_q are maximal by construction, if $t \in D_q$, then $t + s \in D_p$, so $\Phi(t, \Phi(s, p))$ is defined as soon as $\Phi(t+s, p)$ is defined.

Consider Claims (3) and (4). Let $V \subseteq D(\xi)$ denote the subset given by the point (t, p) such that there exist an open interval $J \subseteq \mathbf{R}$ containing 0 and t, and an open neighborhood $p \in U \subseteq X$ such that Φ is defined and smooth on $J \times U \subseteq \mathbf{R} \times X$. By definition, V is open in $\mathbf{R} \times X$, and $\Phi|_V$ is smooth, so it suffices to show that $V = D(\xi)$.

Let $(t_0, p_0) \in D(\xi) - V$. Suppose that $t_0 > 0$. The same argument will apply in the alternative case $t_0 < 0$. Let

$$\tau := \sup\{t \in \mathbf{R} \mid (t, p_0) \in V\}.$$

By Theorem 5.2.18, Φ is defined and smooth in an open neighborhood of $(0, p_0)$, so $\tau > 0$.

Let $q_0 := \Phi_{p_0}(\tau)$. By Theorem 5.2.18, there exist $\varepsilon \in \mathbf{R}_{>0}$ and an open neighborhood $q_0 \in U_0 \subseteq X$ such that $\Phi\colon (-\varepsilon, \varepsilon) \times U_0 \to X$ is defined and smooth.

Choose $t_1 < \tau$ such that $t_1 + \varepsilon > \tau$ and $\Phi_{p_0}(t_1) \in U_0$. By definition of V and $t_1 < \tau$, we have $(t, p_0) \in V$. Thus, Φ is defined and smooth on some $(-\delta, t + \delta) \times U_1$ with $p_0 \in U_1$ open. As $\Phi(t_1, p_0) \in U_0$, there is an open neighborhood $p_0 \in U_1 \subseteq X$ such that $\Phi(\{t_1\} \times U_1) \subseteq U_0$. We have $t_1, p(=)\Phi(t - t_1, \Phi(t_1, p))$ when the right-hand side is defined. By construction of t_1, $\Phi(t_1, p)$ is defined for $p \in U_1$ and smooth in p. As $\Phi(t_1, p) \in U_0$, it follows that $\Phi(t - t_1, \Phi(t_1, p))$ is defined for $p \in U_1$ and $|t - t_1| < \varepsilon$ and smooth in (t, p). This shows that Φ extends to a local flow on $(-\delta, t_1 + \varepsilon) \times U_1$. As $t_1 + \varepsilon > \tau$, this contradicts our choice of τ, so $V = D(\xi)$. □

5.3 Ehresmann's fibration theorem

Definition 5.3.1. Let $f\colon X \to Y$ be a smooth map of smooth manifolds.

(1) We say that f is a *trivial fibration* if there exist a smooth manifold Z and a diffeomorphism $\varphi\colon X \to Y \times Z$ such that $\pi \circ \varphi = f$, where $\pi\colon Y \times Z \to Y$ is the projection. We refer to Z as the *fiber of* f.

(2) We say that f is a *locally trivial fibration* if, for each $y \in Y$, there exists an open neighborhood $V \subseteq Y$ such that $f^{-1}(V) \to V$ is a trivial fibration.

Example 5.3.2. The projection $(x_1, \ldots, x_{m+n}) \mapsto (x_1, \ldots, x_m)\colon \mathbf{R}^{m+n} \to \mathbf{R}^m$ is a trivial fibration.

Remark 5.3.3. By Example 5.1.4, each submersion is a trivial fibration of the form Example 5.3.2 locally on X and Y. In general, however, a submersion $f\colon X \to Y$ need not even be locally trivial fibrations globally on X.

Example 5.3.4. Each map $f\colon \varnothing \to Y$ is a trivial fibration. Indeed, $\varnothing \times Y \simeq \varnothing$.

Example 5.3.5. The 2-torus $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$ is the total space of a trivial fibration over $\mathbf{S}^1$ by projection onto either factor.

Example 5.3.6. The open Möbius strip M is obtained from $X = [0,1] \times (0,1)$ by identifying $(0, t)$ with $(1, 1-t)$ and giving it the quotient topology. Similarly, we may construct $\mathbf{S}^1$ from $[0,1]$ by identifying 0 and 1. Consider the map $\pi\colon X \to [0,1]$ given by $(x, t) \mapsto x$. This map is compatible with the equivalence relations defining M and $\mathbf{S}^1$, so we have a commutative square of continuous maps

$$\begin{array}{ccc} X & \xrightarrow{q} & M \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle f} \\ [0,1] & \xrightarrow{q'} & \mathbf{S}^1 \end{array}$$

in which q and q' are the quotient maps. The map f is a locally trivial fibration The map f is a locally trivial fibration, but it is not a global trivial fibration: if it were, then the total space would be homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^1 = \mathbf{T}^2$. However, the total space M is not orientable, whereas $\mathbf{T}^2$ is orientable. We will discuss orientability later.

Lemma 5.3.7. *Let X be a smooth manifold, let $K \subseteq X$ be a quasi-compact subspace, let $\Phi\colon D \to X$ be a maximal local flow, and let $p \in X$. If $D_p = (a, b) \subseteq \mathbf{R}$ with $b < \infty$, then there exists $\varepsilon \in \mathbf{R}_{>0}$ such that $\Phi(t, p) \notin K$ for $t > b_p - \varepsilon$.*

Proof. Choose an open cover of $\{0\} \times K$ by subsets of $\mathbf{R} \times X$ of the form $V_\alpha := (-\delta_\alpha, \delta_\alpha) \times U_\alpha$ with $\delta_\alpha \in \mathbf{R}_{>0}$,$U_\alpha \subseteq X$ open, and $\alpha \in A$. As $K \simeq \{0\} \times K$ is quasi-compact, there exists a finite subset $A' \subseteq A$ such that the V_α with $\alpha \in A'$ cover $\{0\} \times K$. Letting $0 < \varepsilon < \min\{\delta_\alpha\}_{\alpha \in A'}$, we have $[-\varepsilon, \varepsilon] \times K \subseteq D \cap (\mathbf{R} \times K)$.

Choose $T \in (b - \varepsilon, b)$ with $\Phi(T, p) \in K$ and let $\Phi(t, p) := \Phi(t - T, \Phi(T, p))$ for $T \leq t \leq T + \varepsilon$. This extends Φ to a larger flow domain, as $b < T + \varepsilon$, contradicting the maximality of Φ. □

Lemma 5.3.8. *Let $n \in \mathbf{Z}_{\geq 0}$ and let $f\colon X \to \mathbf{R}^n$ be a smooth map of smooth manifolds. If f is a proper submersion, then f is a trivial fibration.*

Proof. By Corollary 5.1.8, f is an open map. By Proposition 1.5.10.(2), f is also a closed map. Thus, $f(X) \subseteq \mathbf{R}^n$ is both open and closed. By [add reference], $\mathbf{R}^n$ is connected, so it follows that $f(X) = \mathbf{R}^n$ or $X = \varnothing$.

By Example 5.3.4, if $X = \varnothing$, then f is a trivial fibration.

Suppose that $X \neq \varnothing$. As f is a submersion, $m := \dim(X) - n \geq 0$. By Proposition 5.1.1 and Theorem 4.1.19, there exist:

- an open cover $\{U_\alpha\}_{\alpha \in A}$ of X;
- for each $\alpha \in A$, a smooth chart $\varphi_\alpha\colon U_\alpha \to \tilde{U}_\alpha$ with $\tilde{U}_\alpha \subseteq \mathbf{R}^{n+m}$ an open subspace;
- for each $\alpha \in A$, a commutative diagram

(5.3.8.a)
$$\begin{array}{ccccc} U_\alpha & \xleftarrow{\varphi_\alpha^{-1}} & \tilde{U}_\alpha & \hookrightarrow & \mathbf{R}^{m+n} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \mathbf{R}^n & = & \mathbf{R}^n & = & \mathbf{R}^n \end{array}$$

  where π denotes the projection $\pi(x_1, \ldots, x_{m+n}) = (x_1, \ldots, x_n)$; and
- a partition of unity $\{\rho_k\}_{k \in \mathbf{Z}_{\geq 0}}$ subordinate to the cover $\{U_\alpha\}_{\alpha \in A}$.

We introduce the following notation:

- for each $k \in \mathbf{Z}_{\geq 0}$, we choose $\alpha \in A$ such that $\operatorname{supp}(\rho_k) \subseteq U_\alpha$ and we let $\varphi_k := \varphi_\alpha$;
- for each $N \in \mathbf{Z}_{\geq 0}$ and each $1 \leq i \leq N$, e_i denotes the ith standard basis vector in $\mathbf{R}^N$;
- for each $N \in \mathbf{Z}_{\geq 0}$ and each $1 \leq i \leq N$, $D_{i,N}\colon \mathbf{R}^N \to T\mathbf{R}^N$ denotes the smooth vector field that sends each $q \in \mathbf{R}^N$ to the element of $T_q\mathbf{R}^N \simeq \mathbf{R}^N$ given by equivalence class of the path $t \mapsto q + e_i t\colon (-\varepsilon, \varepsilon) \to \mathbf{R}^N$ through q; and
- for each $1 \leq i \leq n$, $\xi_i\colon X \to TX$ denotes the smooth vector field given by

$$\xi_i(q) := \sum_k \rho_k(q) \mathrm{d}(\varphi_k^{-1})(D_{i,m+n}(\varphi_k(q))) = \sum_k \rho_k(q) \cdot [t \mapsto \varphi_k^{-1}(\varphi_k(q) + e_i t)].$$

For each $\alpha \in A$, each $1 \leq i \leq n$, and each $q \in U_\alpha$, we have

(5.3.8.b)
$$\mathrm{d}\pi(D_{i,m+n}(\varphi_\alpha(q))) = D_{i,n}(f(q)).$$

Indeed, this follows from the identifications

$$\begin{aligned} \mathrm{d}\pi(D_{i,m+n}(\varphi_\alpha(q))) &= \mathrm{d}\pi([t \mapsto \varphi_\alpha(q) + e_i t]) \\ &= [t \mapsto \pi(\varphi_\alpha(q) + e_i t)] \\ &= [t \mapsto \pi(\varphi_\alpha(q)) + \pi(e_i t)] \\ &= [t \mapsto f(q) + e_i t] \\ &= D_{i,n}(f(q)), \end{aligned}$$

where we appeal to (5.3.8.a).

We claim that, for each $1 \le i \le n$, the vector field ξ_i lifts the vector field $D_{i,n}$, i.e., the square

$$\begin{CD} X @>{\xi_i}>> TX \\ @V{f}VV @VV{\mathrm{d}f}V \\ \mathbf{R}^n @>{D_{i,n}}>> T\mathbf{R}^n \end{CD} \tag{5.3.8.c}$$

commutes. Indeed, for each $q \in X$, we have

$$\begin{aligned} \mathrm{d}f(\xi_i(q)) &= \mathrm{d}f\Big(\sum_k \rho_k(q)\mathrm{d}(\varphi_k^{-1})(D_{i,m+n}(\varphi_k(q)))\Big) \\ &= \sum_k \rho_k(q)\mathrm{d}f(\mathrm{d}(\varphi_k^{-1})(D_{i,m+n}(\varphi_k(q)))) && \text{[add reference]} \\ &= \sum_k \rho_k(q)\mathrm{d}(f \circ \varphi_k^{-1})(D_{i,m+n}(\varphi_k(q))) \\ &= \sum_k \rho_k(q)\mathrm{d}\pi(D_{i,m+n}(\varphi_k(q))) && (5.3.8.a) \\ &= \sum_k \rho_k(q)D_{i,n}(f(q)) && (5.3.8.b) \\ &= D_{i,n}(f(q)), \end{aligned}$$

where the last equality follows from the definition of a partition of unity.

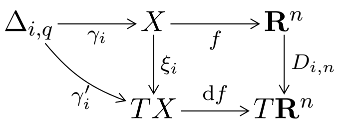
Fix an integer $1 \le i \le n$ and a point $u \in \mathbf{R}^n$. The curve $\bar{\gamma}_i(t) \coloneqq u + e_i t$ is the unique solution to the initial-value problem

$$\bar{\gamma}_i'(t) = e_i, \qquad \bar{\gamma}_i(t) = u,$$

which exists by Theorem 5.2.18. Let $\Phi_i \colon \Delta_i \to X$ be the maximal local flow generated by the smooth vector field ξ_i, as constructed in Theorem 5.2.20. For each $q \in X$, consider the set

$$\Delta_{i,q} \coloneqq \{t \in \mathbf{R} \mid (t, q) \in \Delta_i\} \subseteq \mathbf{R}$$

and the map $\gamma_i \colon \Delta_{i,q} \to X$ given by $\gamma_i(t) \coloneqq \Phi_i(t, q)$. By (5.3.8.c), we have a commutative diagram



Note that $\mathrm{d}f \circ \gamma_i' = \partial(f \circ \gamma_i)/\partial t$ and $(f \circ \gamma_i)(0) = f(q)$. By the uniqueness assertion of Theorem 5.2.18, we must therefore have

$$f(\Phi_i(t, q)) = f(\gamma_i(t)) = f(q) + e_i t \tag{5.3.8.d}$$

for each $t \in \Delta_{i,q}$. In other words, the local flow Φ_i lifts the local flow generated by $D_{i,n}$.

We claim that Φ_i is a global flow. It follows from (5.3.8.d) that, for each $a, b \in \mathbf{R}$, $f(\Phi_i((a,b), q)) \subseteq K$ for some quasi-compact subspace K. For such a subspace K, we have $\Phi_i((a,b), q) \subseteq f^{-1}(K)$. As f is proper by hypothesis, $f^{-1}(K)$ is also quasi-compact. By Lemma 5.3.7, it follows that

$\Delta_{i,q} = \mathbf{R}$: otherwise, the image of $\Phi_i(-, q)$ would escape each quasi-compact subspace in a finite amount of time. This proves the claim that Φ_i is a global flow.

To complete the proof, we use the global flows $\Phi_1, \ldots, \Phi_n$ to construct a diffeomorphism $X \rightsquigarrow \mathbf{R}^n \times f^{-1}(0)$. Consider the map $\tau \colon \mathbf{R}^n \times f^{-1}(0) \to X$ given by

$$\tau(t, q) \coloneqq ((\Phi_1)_{t_1} \circ (\Phi_2)_{t_2} \circ \cdots \circ (\Phi_n)_{t_n})(q),$$

for each $t = (t_1, \ldots, t_n) \in \mathbf{R}^n$ and each $q \in f^{-1}(0)$. This is certainly a smooth map. It admits a smooth inverse $\sigma \colon X \to \mathbf{R}^n \times f^{-1}(0)$ given by

$$\sigma(q) \coloneqq (f(q), ((\Phi_n)_{-f_n(q)} \circ \cdots \circ (\Phi_1)_{-f_1(q)})(q)),$$

which completes the proof. □

Theorem 5.3.9 (Ehresmann)**.** *If $f \colon X \to Y$ is a proper submersion of smooth manifolds, then f is a locally trivial fibration.*

Proof. Without loss of generality, we may assume that $Y = \mathbf{R}^n$. Indeed, the property of being a local trivial fibration is local on the target, so we may cover Y be the domains of its smooth charts, and we may cover each such domain by subspaces diffeomorphic to open disks in $\mathbf{R}^n$, and each such disk is diffeomorphic to $\mathbf{R}^n$ by [add reference]. The claim now follows from Lemma 5.3.8. □

References

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