Topology of algebraic varieties

Brad Drew and Annette Huber

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4 Manifolds and vector bundles

4.1 Manifolds

Definition 4.1.1. A topological space X is called a *topological manifold* of dimension n if

(1) every point has a neighbourhood homeomorphic to an open ball in $\mathbb{R}^n$,

(2) it is Hausdorff,

(3) and *second countable*, i.e., there is countable set of open subsets such that every subset is union of a family of these.

Example 4.1.2. $\mathbb{R}^n$ satisfies all these conditions. We can use open balls $\mathbb{D}(x, r)$ with $x \in \mathbb{Q}^n$ and $r \in \mathbb{Q}_{>0}$ as a basis of the topology.

Remark 4.1.3. Voisin omits the second two conditions. Non-Hausdorff spaces are really bad and we do not want them. Second countability is used to deduce the existence of partitions of unity later on. That's a very useful tool. Our main interest is in affine and projective varieties and they satisfy the property. There is also the notion of a manifold with boundary. In this terminology what we have defined above is also called *manifold without boundary*.

Definition 4.1.4. Let X be a topological manifold An *real n-dimensional chart* (U, φ) *on* X consists of an open subspace $U \subseteq X$, an open subspace $V \subseteq \mathbb{R}^n$, and a homeomorphism $\varphi : U \to V$. Analogously for a *complex n-dimensional chart* (U, φ) we have $V \subset \mathbb{C}^n$.

Example 4.1.5. If $X \subset \mathbb{R}^n$ is open, the identity defines a chart on all of X.

Example 4.1.6. Let $X = S^1 = \{z \in \mathbb{C} | |z| = 1\}$ be the unit circle, $z_0 \in S^1$. We write $z_0 = e^{it_0}$. The map $\pi : \mathbb{R} \to S^1$ mapping t to e^{it} is smooth and surjective, but not injective. It becomes injective when restricting to $V = (t_0 - \epsilon, t_0 + \epsilon)$ for ϵ small enough (e.g., $\epsilon < 2\pi$). Let U be its image under π. Then $\pi|_V : V \to U$ is bijective and in fact a homeomorphism. Its inverse is a chart.

Example 4.1.7. Let $X = \mathbb{R}P^n$. For $i = 0, 1 \dots, n$ let $U_i = \{[x_0 : \cdots : x_n] | x_i \neq 0\}$. Then $\varphi_i : U_i \to \mathbb{R}^n$ defined by

$$\varphi_i([x_0 : \cdots : x_n]) = \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots\right)$$

is a chart. We call these charts *standard charts* of $\mathbb{R}P^n$.

The same construction also works for $\mathbb{C}P^n$.

Exercise 4.1.8. Let $H \subset \mathbb{R}P^n$ by a hyperplane. Construct a chart for $U = \mathbb{R}P^n \smallsetminus H$.

Example 4.1.9. We consider the situation of the implicit function theorem. In the formulation of Forster, Analysis 2: Let $U_1 \subset \mathbb{R}^n$, $U_2 \subset \mathbb{R}^m$ an open. Let

$$F : U_1 \times U_2 \to \mathbb{R}^m, (x, y) \mapsto F(x, y)$$

be smooth. Let $(a, b) \in U_1 \times U_2$ such $F(a, b) = 0$ and $\frac{\partial F}{\partial y}(a, b)$ invertible. Then there are open neighbourhoods $V_1 \subset U_1$ of a and $V_2 \subset U_2$ of b and a C^∞-map

$$g : V_1 \to V_2$$

such that $F(x, g(x)) = 0$ for all $x \in V_1$. Moreover, if $(x, y) \in V_1 \times V_2$ with $F(x, y) = 0$, then $y = g(x)$.

Let $X = \{(x, y) | F(x, y) = 0\} \subset U_1 \times U_2$. Then $U = X \cap V_1 \times V_2$ and the projection $\varphi : U \to V_1$ is a chart. The inverse map is given by $x \mapsto (x, g(x))$.

Definition 4.1.10. A *smooth atlas* $(U_i, \varphi_i)_{i\in I}$ *on* X is a family of real n-dimensional charts $(U_i, \varphi_i)_{i\in I}$ on X satisfying the following conditions:

(1) $(U_i)_{i\in I}$ is an open cover of X; and
(2) for each $i, j \in I$, the *transition map*

$$\varphi_j(U_i \cap U_j) \xrightarrow{\varphi_i^{-1}} U_i \cap U_j \xrightarrow{\varphi_j} \varphi_j(U_i \cap U_j)$$

is a $\mathcal{C}^\infty$-map (Definition 2.2.3).

We say that two atlases $(U_i, \varphi_i)_{i\in I}$ and $(U'_j, \varphi'_j)_{j\in J}$ on X are *equivalent* if their union is an atlas. Analogously we define a *holomorphic atlas* $(U_i, \varphi_i)_{i\in I}$ using complex charts and the transition maps are holomorphic.

A *smooth manifold* is a topological manifold X together with the choice of an equivalence class of smooth atlases. Analogously a *complex manifold* is a topological manifold X together with the choice of an equivalence class of holomorphic atlases. A complex manifold of dimension 1 is also called *Riemann surface*

Example 4.1.11. $U \subseteq \mathbf{R}^n$, $\mathbf{S}^n$, $\mathbf{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, Möbius strip, Klein bottle, $\mathbf{RP}^n$, $y^2 = x^3 + x^2$

Lemma 4.1.12. *Let $U \subset \mathbb{R}^N$ open. Let*

$$F : U \to \mathbb{R}^m$$

be smooth. Assume that the Jacobian $\left(\frac{\partial F_i}{\partial x_j}\right)_{i,j}$ has rank $N - m$ in all points of $X = \{(x \in \mathbb{R}^N \mid F(x) = 0\}$. Then X carries a canonical smooth-manifold structure.

Proof. Let $x \in X$. After reordering of the coordinates of $\mathbb{R}^N$ we may choose a neighbourhood of x in $\mathbb{R}^N$ of the form $U_1 \times U_2 \subset \mathbb{R}^{N-m} \times \mathbb{R}^m$ satisfying the assumptions of the implicit function theorem. This defines a chart in a neighbourhood of x. The transition maps are expressed in terms of the function g defined in the implicit function theorem. In particular it is smooth. We have found an atlas. □

The same lemma also works in the holomorphic setting.

Example 4.1.13. $X = \{(x,y) \in \mathbb{C}^2 | y^2 = x^3 + x^2\}$. We apply the lemma with $F(x,y) = y^2 - x^3 - x^2$. The Jacobi matrix is the gradient

$$\Big(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\Big) = (-3x^2 - 2x, 2y).$$

It has maximal rank (i.e., 1) unless $y = 0$ and $3x^2 + 2x = x(3x+2) = 0$. Note that $(-2/3, 0)$ does not satisfy F, hence there is only one problematic point $(0,0)$. We get a manifold structure on $X^\circ = X \smallsetminus \{(0,0)\}$.

We call $(0,0)$ a *singularity* of X. We will study them in more detail later on.

Example 4.1.14. Every holomorphic manifold of complex dimension n also defines a smooth manifold of real dimension $2n$ by identifying $\mathbb{C}^n \rightsquigarrow \mathbb{R}^{2n}$. It is an interesting question, but not important to us, which smooth manifolds can be given the structure of a complex manifold.

The next step is to define morphisms, so that we get categories.

Definition 4.1.15. Let X and Y be smooth (or holomorphic) manifolds and let $f : X \to Y$ be a map. We say that f is smooth (or holomorphic) if, for each $x \in X$, there exists a chart (U, φ) with $x \in U$ and a chart (V, ψ) with $f(x) \in V$ such that $f(U) \subseteq V$ and the composite

$$\varphi(U) \xrightarrow{\varphi^{-1}} U \xrightarrow{f} V \xrightarrow{\psi} \psi(V)$$

is smooth (or holomorphic). We say that f is a *smooth diffeomorphism* if it is a bijective smooth map. We say that f is an *isomorphism of complex manifolds* if it is bijective and holomorphic.

Exercise 4.1.16. Show that the inverse f^{-1} of a smooth diffeomorphism is also smooth and that the inverse f^{-1} of an isomorphism of complex manifolds is holomorphic.

Proposition 4.1.17. *Let $f : X \to Y$ be a smooth map of smooth manifolds, let $x \in X$, let (U, φ) be a chart with $x \in U$, and let (V, ψ) be a chart with $f(x) \in V$ such that $f(U) \subseteq V$. The composite*

$$\varphi(U) \xrightarrow{\varphi^{-1}} U \xrightarrow{f} V \xrightarrow{\psi} \psi(V)$$

is a smooth map.

Proof. Exercise. Note that smoothness of functions $U \to \mathbb{R}$ for $U \subset \mathbb{R}^n$ is a local property. ☐

Exercise 4.1.18. Let $U \subset \mathbb{C}$, $f : U \to \mathbb{C}$ holomorphic viewed as a holomorphic map of complex manifolds. Let $z_0 \in U$. Show that there are charts (U_1, φ_1) around z_0 with $\varphi_1(z_0) = 0$ and (U_2, φ_2) around $f(z_0)$ with $\varphi_2(f(z_0)) = 0$ such that $f(U_1) \subset U_2$ and the induced map

$$V_1 \xrightarrow{\varphi_1^{-1}} U_1 \to U_2 \xrightarrow{\varphi_2} V_2$$

is given by $z \mapsto z^d$ for a natural number d.

Theorem 4.1.19 (Partition of Unity)**.** *Let X be a smooth manifold, $\{U_\alpha\}_{\alpha \in I}$ an open cover. Then there are smooth functions $f_n : X \to \mathbb{R}$ for $n \in \mathbb{N}$ with compact support taking values in $[0,1]$ such that:*

1. *for every n there is $\alpha \in I$ such that the support of f_i is contained in U_α;*
2. *for every $x \in X$ there are only finitely many α such that $f_\alpha(x) \neq 0$;*
3. $\sum_\alpha f_\alpha = 1$.

Example 4.1.20. Let $X = \mathbb{R}$ covered by a single $U = \mathbb{R}$. Putting $f_1 = 1$ does not do the trick because we want functions with compact support. Instead choose smooth positive functions g_n ($n \in \mathbb{Z}$) such that $g_n(x) = 0$ for $x \notin [n-1, n+2]$, $g_n(x) > 0$ on $[n, n+1]$. They have compact support and for every $x \in \mathbb{R}$ there are only finitely many n with $g_n(x) \neq 0$. Hence $g = \sum g_n$ exists. We put for $n \in \mathbb{N}_0$

$$f_n = \frac{g_n + g_{-n}}{g}.$$

Then the functions sum to 1. They are all positive, so the values are in $[0, 1]$.

The argument is fiddly and we do not give it here. Let us just see how the manifold axioms come in. For all other details see [War83, Theorem 1.11].

Lemma 4.1.21 ([War83, Lemma 1.9])**.** *Let X be a topological space which is locally compact, Hausdorff and second countable. Then every open cover has a countable, locally finite refinement consisting only of open sets with compact closures.*

Proof. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ be a fixed open cover. As X is second countable, there is a sequence $\{V_i\}_{i \in \mathbb{N}}$ of open sets such that every open set is a union of some of the V_i. For every $x \in X$ we choose a $i(x) \in \mathbb{N}$ such that $x \in V_i$, $\bar{V}_i$ is compact. The compactness condition is achieved by picking V_i such that there is a chart (U, φ) with $V_i \subset U$ and $\varphi(V_i)$ contained in a small enough open ball that the closure is still in $\varphi(U)$. The family $\{V_{i(x)}\}_{\{i(x) | x \in X\}}$ is a countable refinement of $\mathfrak{V}$. Its elements have a compact closure. We replace $\mathfrak{V}$ by this subcover.

We construct a sequence

$$G_1 \subset G_2 \subset \dots$$

of open subsets with compact closure that $\bigcup_{i=1}^{\infty} = X$ and

$$\bar{G}_i \subset G_{i+1}.$$

Put $G_1 = V_1$. Let j_2 be the smallest index greater than 1 such that

$$\bar{G}_1 \subset \bigcup_{i=1}^{j_2} U_i.$$

Such an index exists because $\bar{G}_1$ is compact, hence the cover by all of $\mathfrak{V}$ has a finite subcover. We put

$$G_2 = \bigcup_{i=1}^{j_2} V_i.$$

Let j_3 be the smallest index greater than j_2 such that

$$\bar{G}_2 \subset \bigcup i = 1^{j_2} U_i = G_3.$$

This sequence has the properties we wanted for G_i.

Now let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ be a fixed open cover. The set $\bar{G}_i \setminus \bar{G}_{i-1}$ is compact and contained in the open set $G_{i+1} \setminus \bar{G}_{i-2}$. For each $i \geq 3$ choose a finite subcover of the open cover $\{U_\alpha \cap (G_{i+1} \setminus \bar{G}_{i-2}\}_\alpha$. Also choose a finite subcover of the open cover $\{U_\alpha \cap \bar{G}_2\}$. This collection of open sets does the trick. ☐

Idea Proof of Theorem 4.1.19. A partition of unity subordinate to a refinement is also subordinate to the original cover. Hence it suffices to consider a countable, locally finite cover such that $\bar{U}_\alpha$ is compact. This looks very much like the cover of $\mathbb{R}$ by intervals $(n-1, n+2)$ that we used in the example. After passing to coordinate charts, the same type of functions as in the example can be used on balls. ☐

4.2 Vector bundles

Our next aim is to define the *tangent bundle* of a real or complex manifold. We start with an example:

Example 4.2.1. We consider the unit circle $S^1 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$. The *tangent line* at (x_0, y_0) is has the parametrisation

$$t \mapsto (y_0, -x_0)t + (x_0, y_0).$$

The *tangent space* is given by

$$T_{(x_0,y_0)}S^1 = \{t(y_0, -x_0) | t \in \mathbb{R}\}.$$

It is a vector space of the same dimension as S^1. This space varies as (x_0, y_0) varies. Together they form the *tangent bundle*

$$TS^1 = \{(v, w) \in S^1 \times \mathbb{R}^2 | v \perp w.\}$$

It comes with a projection $p : TS^1 \to S^1$. The fibre of v has the structure of a vector space.

The first step is to define objects like the tangent bundle.

Definition 4.2.2. Let X be a topological space. A *real vector bundle of rank r on X* is a consists of a continuous

$$\pi : V \to X$$

together with the structure of a real vector space of dimension $\mathbb{R}$ on $V_x = \pi^{-1}(x)$ such that there exists an open cover an open cover $(U_i)_{i \in I}$ of X, and *local trivialisations*, i.e., homeomorphisms

$$\tau_i : \pi^{-1}(U_i) \to U_i \times \mathbf{R}^r$$

satisfying the following conditions:

(1) for each $i \in I$, the triangle

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\tau_i} & U_i \times \mathbf{R}^r \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle p} & \\ & U_i & \end{array}$$

commutes, where $p : U_i \times \mathbf{R}^r \to U_i$ denotes the projection; and

(2) for every $i \in I$, $x \in U_i$, the induced map

$$V_x \to \{x\} \times \mathbb{R}^r \to \mathbb{R}^r$$

is an isomorphism of $\mathbb{R}$-vector spaces.

Let $f : X \to Y$ be continuous, $\pi : V \to X$ and $\xi : W \to Y$ real vector bundles. A *morphism of vector bundles over f* is a continuous map $F : V \to W$ making the diagram

$$\begin{array}{ccc} V & \xrightarrow{F} & W \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

commute such that for all $x \in X$ the induced map

$$F_x : V_x \to W_{f(x)}$$

is $\mathbb{R}$-linear.

If we replace $\mathbb{R}$ by $\mathbb{C}$ in this definition, we get the notion of a *complex vector bundle of rank r*. A real or complex vector bundle of rank 1 is called a real or complex *line bundle* (Geradenbündel in German).

Remark 4.2.3. By identifying $\mathbb{C}$ with $\mathbb{R}^2$, a complex vector bundle of rank r also defines a real vector bundle of rank $2r$.

Exercise 4.2.4. Let $\pi : V \to X$ be a vector bundle of rank r. Show that it is *trivial*, i.e., isomorphic to $X \times \mathbb{R}^r$ with the standard vector bundle structure if and only if there are r continuous sections $e_i : X \to V$ (i.e., maps satisfying $\pi \circ e_i = \mathrm{id}_X$) such that for every $x \in X$ the tuple $e_1(x), \ldots, e_r(x)$ is a basis of V_x.

Given a local trivialisation, the induced maps τ_{ij} for each $i, j \in I$ in the commutative diagram

$$\begin{array}{ccccc} \tau_i(\pi^{-1}(U_i \cap U_j)) & \xrightarrow{\tau_i^{-1}} & \pi^{-1}(U_i \cap U_j) & \xrightarrow{\tau_j} & \tau_j(\pi^{-1}(U_i \cap U_j)) \\ \sim\downarrow & & & & \sim\updownarrow \\ (U_i \cap U_j) \times \mathbf{R}^r & & \xrightarrow{\tau_{ij}} & & (U_i \cap U_j) \times \mathbf{R}^r \end{array}$$

are called *transition maps.* They are automatically continuous, respect the fibres over $U_i \cap U_j$ and are fibrewise linear. Hence the data of τ_{ij} is equivalent to the data of a continuous map

$$\varphi_{ij} : U_i \cap U_j \to \mathrm{Gl}_r(\mathbb{R}).$$

The transition maps are not arbitrary, but have to be compatible on triple intersections. They satisfy the *cocycle condition*

$$\varphi_{jk} \cdot \varphi_{ij} = \varphi_{ik}.$$

Exercise 4.2.5. The cocycle condition implies $\varphi_{ii} = \mathrm{id}$.

Proposition 4.2.6. *Let X be topological space, $\mathfrak{U} = (U_i)_{i \in I}$ an open cover and*

$$\varphi_{ij} : U_i \cap U_j \to \mathrm{Gl}_r(\mathbb{R})$$

continuous such that the cocycle condition is satisfied on $U_i \cap U_j \cap U_k$ for all $i, j, k \in I$. Then there is a real vector bundle $\pi : V \to X$ of rank r with transition functions

$$\tau_{ij} : U_i \cap U_j \times \mathbb{R}^r \to U_i \cap U_j \times \mathbb{R}^r$$

given by

$$(x, v) \mapsto (x, \varphi_{ij}(x)(v)).$$

The bundle V is unique up to unique isomorphism.

Proof. We glue $V_i = U_i \times \mathbb{R}^r$ along the open subsets $U_i \cap U_j \times \mathbb{R}^r$ inside V_i and V_j using τ_{ij} as gluing maps. This is well-defined because of the cocycle condition. The gluing gives a topological space V. By construction, there is a continuous map $V \to X$, the fibres are given vector space structures and we have local trivialisations. □

Example 4.2.7. Let $X = S^1$, covered by $U_1 = \{(x, y) \in S^1 | x \neq -1\}$ and $U_2 = \{(x, y) \in S^1 | x \neq -1\}$. The trivial line bundle $S^1 \times \mathbb{R}$ has the transition map

$$\varphi_{12}(v) = 1 \in \mathrm{Gl}_r(\mathbb{R}).$$

There is also the Möbius bundle with transition map

$$\varphi_{12}(v) = -1.$$

All construction for vector spaces (direct sums, tensor products, exterior powers, ...) can also applied to vector bundles. For example:

Lemma 4.2.8. *Let X be a topological space, $\pi : V \to X$ a vector bundle of rank r. Then there is a canonical vector bundle $\xi : V^* \to X$, the* dual bundle *of V, with $\xi^{-1}(x) = V^*$.*

Proof. We use local trivialisations on a cover $(U_i)_{i\in I}$ and transition maps $\varphi_{ij} : U_i \cap U_j \to \mathbb{R}^r$ for V. Let $\psi_{ij} = (\varphi_{ij}^{-1})^t$. It is easy to check that they satisfy the cocycle condition and hence define a vector. This is V^*. □

Exercise 4.2.9. Write out all diagrams!

Definition 4.2.10. (1) Let X be a smooth manifold. A *smooth* real or complex vector bundle is a smooth map

$$\pi : V \to X$$

of smooth manifolds together with the structure of a real or complex vector bundle on π such that the local trivialisations are smooth maps.

(2) Let X be a complex manifold. A *holomorphic* vector bundle is a holomorphic map

$$\pi : V \to X$$

of complex manifolds together with the structure of a complex vector bundle on π such that the local trivialisations are holomorphic.

In the case of a smooth vector bundle, the transition maps into $\mathrm{Gl}_r(\mathbb{R})$ or $\mathrm{Gl}_r(\mathbb{C})$ are smooth. In the case of a holomorphic vector bundle, the transition maps are holomorphic.

Remark 4.2.11. Given a holomorphic vector bundle of rank r on a complex manifold, we may also view it as a smooth complex vector bundle on the associated real manifold.

4.3 Tangent bundles

Let X be a real manifold of dimension n, $x \in X$. We want to define the tangent space of X in x. Tangent vectors are directions on X.

Example 4.3.1. If $X \subset \mathbb{R}^n$ is open, then the tangent space is $\mathbb{R}^n$. Every vector $v \in \mathbb{R}^n$ defines a straight line

$$t \mapsto x + tv$$

in X.

In general, it does make sense to talk about straight lines in X. If (U, φ) is a chart around x, we could use the preimages of straight lines in $\varphi(U)$. However, the result will depend on the chart. Nevertheless:

Definition 4.3.2. Let X be a smooth manifold, $x \in X$. A *smooth path* through x is a differentiable map

$$\gamma : (-\epsilon, \epsilon) \to X$$

for some $\epsilon > 0$ such that $\gamma(0) = x$. We say that two paths are *equivalent* if they have the same tangent vector in $\mathbb{R}^n$ for some chart (U, φ) around x. A *tangent vector* is an equivalence class of paths. Let T_xX be the set of tangent vectors of X in x.

The composition $\bar{\gamma} = \varphi \circ \gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ is defined, possibly after making ϵ smaller. The tangent vector is given by the formula

$$((\varphi\gamma)_1'(0), \dots, (\varphi\gamma)_n').$$

If (V, ψ) is a second chart around x, then the chain rule expresses the tangent vector in $\psi(V)$ in terms of the Jacobian of the transition morphism and the tangent vector in $\varphi(U)$. In particular, the equivalence relation is independent of the choice of chart.

There is a more elegant approach. Recall that a path defines a *directional derivative*.

Definition 4.3.3. Let X be a smooth manifold, $x \in X$, γ a smooth path through x. Let $f : U \to \mathbb{R}$ be differentiable for some open neighbourhood of X. We put

$$\partial_\gamma f := \lim_{h \to 0} \frac{f(\gamma(0+h)) - f(x)}{h} = (f \circ \gamma)'(0)$$

the *derivative of f in the direction of γ*.

Note that this definition does not mention charts! The differentiability assumptions on f and γ make the composition a differentiable function, so the limit exists.

Lemma 4.3.4. *Two paths define the same tangent vector at x if and only if, for all smooth $f : U \to \mathbb{R}$, the directional derivatives agree.*

Proof. Let γ_1 and γ_2 be equivalent paths. We choose a chart (V, φ) around x. By shrinking U and V, we may assume that f is defined on V. For the rest of the argument, we may replace X by $\varphi(V)$, f by the $f \circ \varphi^{-1}$ and γ_i by $\varphi \circ \gamma_i$. This does not change the directional derivative. Hence without loss of generality $f : U \to \mathbb{R}$ for $U \subset \mathbb{R}^n$. By the chain rule

$$\partial_{\gamma_i} f = \Big(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\Big)((\gamma_i)'_1(0), \dots, (\gamma_i)'_n(0))^t.$$

Hence it only depends on the tangent vector, i.e., the equivalence class.

Conversely, assume that γ_1 and γ_2 define the same directional derivative for all f. Choose a chart (U, φ) around x and consider the coordinate functions $f_j = x_j \circ \varphi$. The tangent vector of $\varphi \circ \gamma_i$ is given by the directional derivatives

$$(\partial_{\gamma_i} f_1, \dots, \partial_{\gamma_i} f_n).$$

Hence γ_1 and γ_2 are equivalent. ☐

So from now on, we think of tangent vectors as directional derivatives. They are maps. But on what?

Definition 4.3.5. Let X be a smooth manifold, $x \in X$. Two smooth functions $f_1 : U_1 \to \mathbb{R}$, $f_2 : U_2 \to \mathbb{R}$ for open neighbourhoods $U_i \subset \mathbb{R}$ of x are called equivalent at x if there is an open neighbourhood $U \subset U_1 \cap U_2$ of x such that

$$f_1|_U = f_2|_U.$$

The equivalence class of $f : U \to \mathbb{R}$ is called the *germ* (Keim) of f at x. We denote the set (even $\mathbb{R}$-algebra) of germs at x by $\mathcal{A}_x$.

Remark 4.3.6. The ring $\mathcal{A}_x$ is local with maximal ideal m_x consisting of the germs of functions f with $f(x) = 0$. All other germs are invertible.

Remark 4.3.7. We can make the same definition for complex manifolds and holomorphic functions. In this case two functions define the same germ if they have the same power series expansion in some chart. The ring of germs is denoted $\mathcal{O}_x$ in this case.

Lemma 4.3.8. *Let X be a smooth manifold, $x \in X$, γ a path through x. Then*

$$\partial_\gamma : \mathcal{A}_x \to \mathbb{R}$$

is well-defined and an $\mathbb{R}$-derivation, *i.e.,* $\mathbb{R}$*-linear and the Leibniz rule holds:*

$$\partial_\gamma(fg) = f(x)\partial_\gamma g + g(x)\partial_\gamma f.$$

Proof. The definition of $f \circ \gamma$ does not change when making the neighbourhoods smaller. The Leibniz rule holds because it holds for ordinary derivatives. ☐

One advantage of this point of view: the set of $\mathbb{R}$-derivations is a vector space!

Proposition 4.3.9. *Let X be a smooth manifold of dimension n, $x \in X$. Then the set of $\mathbb{R}$-derivations is an $\mathbb{R}$-vector space of dimension n. The map that assigns to an equivalence class of paths its directional derivative is bijective.*

Proof. Let (U, φ) be a chart. The claims for U and $\varphi(U)$ are equivalent. Hence we can replace X by an open subset $V \subset \mathbb{R}^n$ and x. Let $x_1, \dots, x_n$ be the coordinate functions on V. The directional derivative for the path $(-\epsilon, \epsilon) \to V$ mapping t to $x + te_i$ is the partial derivative $\partial_i := \frac{\partial}{\partial x_i}$. We claim that $\partial_1, \dots, \partial_n$ is a basis for the space of derivations. This will prove the dimension formula as well as surjectivity.

We start with linear independence. Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that

$$\lambda_1 \partial_1 + \dots \lambda_n \partial_n = 0$$

as maps $\mathcal{A}_x \to \mathbb{R}$. The left hand side is equal to the directional derivative for the tangent vector $\sum \lambda_i e_i$. We have already shown that this implies vanishing of the tangent vector. As $e_1, \dots, e_n$ is basis, this means that all λ_i vanish.

We now need to show that the ∂_i generate the space of derivations. Let $\partial : \mathcal{A}_x \to \mathbb{R}$ be a derivation. Put

$$\lambda_i = \partial x_i.$$

We claim that $\partial = \sum \lambda_i \partial_i$. Let $f : V \to \mathbb{R}$ be smooth. Hence (Taylor expansion)

$$f(y) = f(x) + \sum_j \partial_j f(x)(y_j - x_j) + \sum g_{jk}(y_j - x_j)(y_k - x_k)$$

with smooth functions g_{jk}. We apply ∂ and $\sum(\partial x_i)\partial_i$ to the equation and use linearity and the Leibniz rule. They vanish on the first and last summand and have the same effect on the second summand. ☐

We record further insights from the proof:

Corollary 4.3.10. (1) *$T_x X$ is the dual vector space of m_x/m_x^2.*

(2) *If (U, φ) is a chart, then for all $x \in U$, the directional derivatives $\partial_i = \partial_{\gamma_i}$ with $\gamma_i(t) = \varphi^{-1}(x + te_i)$ are a basis of $T_x X$.*

Definition 4.3.11. Let X be a smooth manifold of dimension n.

(1) We define the *tangent bundle* TX as the set $\coprod_{x \in X} T_x X$ with $\pi(v) = x$ for $v \in T_x X$. For every coordinate chart (U, φ) of X, we use the bijection

$$\pi^{-1}(U) \to U \times \mathbb{R}^d, \quad v = \sum_{i=1}^n \lambda_i(x)\partial_i \mapsto (x, \lambda_1(x), \dots, \lambda_n(x))$$

to define a topology and a smooth structure on TX.

(2) We define the *cotangent bundle* T^*X as the dual vector bundle of TX.

(3) $U \subset X$ open. A section $s : U \to TX$ (i.e., a smooth map satisfying $\pi \circ s = \mathrm{id}_U$) is called *tangent vector field*. A section $t : U \to T^*X$ is called *differential form*. Sections of $\Lambda^r T^*X$ are called *differential forms of degree r*.

Exercise 4.3.12. Let X and Y be smooth manifolds of dimensions m and n, respectively. The product $X \times Y$ admits a unique smooth structure such that the projections $X \times Y \to X$ and $X \times Y \to Y$ are smooth.

Exercise 4.3.13. Let $X \subset \mathbb{R}^{n+1}$ be the manifold defined as the vanishing locus of a function $F : U \to \mathbb{R}$ for $U \subset \mathbb{R}^{n+1}$ open such that the gradient of F is non-zero on X. Make the identification of $T_x X$ with the space of directions tangent to X in $\mathbb{R}^{n+1}$ explicit.

Remark 4.3.14. Our proofs have given a natural identification $T_x^*X = m_x/m_x^2$. If $f : U \to \mathbb{R}$ is smooth for $U \subset X$ open, we get a differential form

$$df : x \mapsto [f - f(x)].$$

Exercise 4.3.15. Check that df is smooth. As a first step show that on a coordinate chart (U, φ) the cotangent vector $d\varphi_i(x)$ is the dual of the standard tangent vector ∂_i.

4.4 Tangent bundles of complex manifolds

All constructions and definitions can be translated from the smooth to the holomorphic setting. We replace paths defined on $(-\epsilon, \epsilon)$ by holomorphic maps defined on an open disc $\mathbb{D}(0, \epsilon)$. They give rise to directional derivatives. They are examples of $\mathbb{C}$-linear derivations. The set of $\mathbb{C}$-linear derivations can be identified with the dual of m_x/m_x^2 where this time $m_x \subset \mathcal{O}_x$. If (U, φ) is a chart, then the standard basis on $\mathbb{C}^n$ gives rise to a basis ∂_i of the tangent space. The tangent bundle $TX \to X$ is a holomorphic vector bundle of rank equal to the complex dimension of X.

Remark 4.4.1. Let X be a complex manifold of dimension n. We denote by $X_{\mathbb{R}}$ the underlying smooth manifold. Then the holomorphic tangent bundle TX has complex dimension n and hence real dimension $2n$. The smooth tangent bundle $TX_{\mathbb{R}}$ has real dimension $2n$ because $X_{\mathbb{R}}$ has dimension $2n$. There is a actually a natural isomorphism of smooth vector bundles

$$(TX)_{\mathbb{R}} \to TX_{\mathbb{R}}.$$

Given $\sigma : \mathbb{D}(0, \epsilon) \to X$, we have the path $\gamma : (-\epsilon, \epsilon) \to \mathbb{D}(0, \epsilon) \to X$. The bundle map is

$$\partial_\sigma \mapsto \partial_\gamma.$$

It is $\mathbb{R}$-linear, but not complex linear.

Exercise 4.4.2. Write down the identification explicitly in the case $X \subset \mathbb{C}$ open.

Remark 4.4.3 (continued)**.** The above identification has to be distinguished from a different construction. Given a real vector bundle $V \to X$ for a smooth manifold X, we also have the complex vector bundle $V \otimes_{\mathbb{R}} \mathbb{C} \to X$. In particular, there is the complexified tangent bundle $TX \otimes_{\mathbb{R}} \mathbb{C}$. If X has real dimension m, then $TX \otimes_{\mathbb{R}} \mathbb{C}$ has complex rank m and real rank $2m$. This can also be applied to the smooth manifold $X_{\mathbb{R}}$ underlying a complex manifold X of dimension n. Hence TX has complex rank n and $TX \otimes_{\mathbb{R}} \mathbb{C}$ has complex rank $2n$.

4.5 Functoriality

Let $\psi : X \to Y$ be a smooth morphism, $x \in X$. We define $d\psi_x \colon T_xX \to T_{\psi(x)}Y$ as follows: fix $\partial \in T_xX$. Let $f \in \mathcal{A}_{\psi(x)}$ be represented by $f : U \to \mathbb{R}$ for $U \subset X$ an open neighbourhood of $\psi(x)$. Then $f \circ \psi : \psi^{-1}U \to \mathbb{R}$ defines an element of $\mathcal{A}_x$. The map

$$f \mapsto \partial(f \circ \psi)$$

is a derivation on $\mathcal{A}_{\psi(x)}$. We have found a linear map as we wanted.

Definition 4.5.1. Let $f : X \to Y$ be a smooth morphism. We call

$$df : TX \to TY$$

the induced morphism on tangent bundles.

Remark 4.5.2. df is indeed a morphism of bundles: linear on fibres and smooth.

Proposition 4.5.3. *The assignments $X \mapsto TX$ and $f \mapsto \mathrm{d}f$ define a functor from the category of smooth manifolds and smooth maps to itself. In other words:*

(1) *for each smooth manifold X, $\mathrm{id}_{TX} = \mathrm{d}(\mathrm{id}_X)\colon TX \to TX$; and*

(2) *if $f\colon X \to Y$ and $g\colon Y \to Z$ are smooth maps of smooth manifolds, then $\mathrm{d}(g \circ f) = \mathrm{d}f \circ \mathrm{d}g$ as maps $TX \to TZ$.*

In particular, if $f\colon X \to Y$ is a diffeomorphism, then $\mathrm{d}f$ is an isomorphism.

Definition 4.5.4. Let $f\colon X \to Y$ be a smooth map of smooth manifolds and let $x \in X$. The *rank* $\mathrm{rk}_x(f)$ *of f at x* is the rank of the $\mathbb{R}$-linear map $\mathrm{d}f_x\colon T_xX \to T_{f(x)}Y$. We say that f is *of constant rank* if $x \mapsto \mathrm{rk}_x(f)\colon X \to \mathbb{Z}_{\geq 0}$ is a constant function.

Remark 4.5.5. The rank $\mathrm{rk}_x(f)$ of a smooth map of smooth manifolds $f\colon X \to Y$ is bounded above by $\min(\dim(X), \dim(Y))$: indeed, $\dim(X) = \dim(T_xX)$, $\dim(Y) = \dim(T_{f(x)}Y)$, and the image of $\mathrm{d}f_x$ is both a quotient of T_xX and a subspace of $T_{f(x)}Y$.

Proposition 4.5.6. *Let $f\colon X \to Y$ be a smooth map of smooth manifolds, let $x \in X$, and let $r := \mathrm{rk}_x(f)$. There exists an open neighborhood $x \in U \subseteq X$ such that $\mathrm{rk}_{x'}(f) \geq r$ for each $x' \in U$.*

Proof. The question is local on X and on Y. We may therefore assume without loss of generality that $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ are open subsets. At each point $x' \in X$, $\mathrm{d}f_x$ is represented by the Jacobian matrix with respect to the standard bases, and the condition that $\mathrm{rk}_{x'}(f) \geq r$ is precisely the condition that the Jacobian matrix of f at x' is of rank $\geq r$.

As f is smooth, its partial derivatives and, hence, its Jacobian matrices vary continuously. Using this continuity, it suffices to show that the set of points $x' \in X$ at which the Jacobian matrix of f is of rank $\geq r$ is open in the set of all $n \times m$ matrices.

To say that an $n \times m$ matrix M is of rank $\geq r$ is to say that there exists an $r \times r$ submatrix M' of M such that $\det(M') \neq 0$. The determinant is a polynomial in the entries of a matrix, so the determinant function is continuous. Thus, the nonvanishing of the determinant is an open condition. The condition that a matrix be of rank $\geq r$ is therefore also an open condition. □

4.6 Immersions, embeddings, and submanifolds

Definition 4.6.1. Let $f : X \to Y$ be a smooth morphism.

(1) f is called an *immersion* if df_x is injective for all $x \in X$.

(2) X is a *submanifold* if f is injective and an immersion.

(3) f is an *embedding* if it is an immersion and f is a homeomorphism onto its image $f(X)$.

There are nasty counterexamples saying that the three conditions are not equivalent. They are a phenomenon of real analysis. In the holomorphic category they do not occur.

Example 4.6.2. Let $m, n \in \mathbb{Z}_{\geq 0}$. The inclusion $x \mapsto (x, 0)\colon \mathbb{R}^m \to \mathbb{R}^{m+n}$ is an embedding. We will see later ([add reference]) that every immersion is locally of this form in a suitable sense.

Exercise 4.6.3. Show that $S^1 \subset \mathbb{R}^2$ is an embedding.

Example 4.6.4. The map $f\colon x \mapsto x^3\colon \mathbb{R} \to \mathbb{R}$ is a homeomorphism. By [add reference], $\mathrm{d}f_x$ is given by multiplication by $f'(x) = 3x^2$, so $\mathrm{d}f_0$ is the zero map. The map f is therefore not an immersion.

Example 4.6.5. Consider the map $f\colon x \mapsto (\cos(x), \sin(x))\colon [0, 2\pi) \to \mathbb{R}^2$. By [add reference], $\mathrm{d}f_x$ is represented by the Jacobian matrix

$$\begin{bmatrix} -\sin(x) \\ \cos(x) \end{bmatrix}.$$

As $\sin(x)$ and $\cos(x)$ do not vanish simultaneously, $\mathrm{d}f_x$ is injective for each $x \in [0, 2\pi)$, and f is therefore an immersion. The image of f is the unit circle $S^1 \subseteq \mathbb{R}^2$, which is a quasi-compact subspace by [add reference]. The domain $[0, 2\pi)$, however, is not quasi-compact by the Heine-Borel Theorem (Theorem 1.3.33). Thus, the map f cannot be a homeomorphism onto its image: quasi-compactness, like all toplogical properties, is homeomorphism invariant.

4.7 Submersions

Definition 4.7.1. A smooth morphism $f \colon X \to Y$ is a *submersion* if $\mathrm{d}f_x$ is surjective for each $x \in X$.

Example 4.7.2. Let $m, n \in \mathbb{Z}_{\geq 0}$. The projection $(x, y) \mapsto x \colon \mathbb{R}^{m+n} \to \mathbb{R}^m$ is a submersion. We will see later ([add reference]) that every submersion is locally of this form in a suitable sense.

Definition 4.7.3. Let X be a topological space. It is a *topological manifold with boundary* (*with corners*) if it is Hausdorff and second countable and every point of x has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ (of $\mathbb{R}^n_{\geq 0}$, respectively).

As in the case without boundary, we have the notion of an equivalence class of atlases by imposing that the transition maps are smooth. Recall that we called a map $A \to B$ between subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ smooth if it extends to a smooth function on an open neighbourhood of A.

Exercise 4.7.4. Let X be a compact manifold with boundary, ω a smooth differential form on X. Look up and understand the Theorem of Stokes:

$$\int_X d\omega = \int_{\partial X} \omega.$$

What does it say for $X = [a, b] \subset \mathbb{R}$?

References

[War83] Frank W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition.