Topology of algebraic varieties

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Contents

3 Holomorphic functions

Motivation. In the previous chapter, we showed that each complex-analytic function is complex differentiable. In this chapter, which is not essential for the ultimate goals of this course, we establish the converse.

3.1 Contour integration

Motivation. Our next goal is to establish the converse of Proposition 2.4.4.

Definition 3.1.1. Let $a, b \in \mathbf{R}$, let $f\colon [a, b] \to \mathbf{C}$ be a continuous function, and let $u = \mathrm{Re} \circ f$ and $v = \mathrm{Im} \circ f$. We define

$$\int_a^b f(t)\,\mathrm{d}t := \int_a^b (u(t) + \mathrm{i}v(t))\,\mathrm{d}t = \int_a^b u(t)\,\mathrm{d}t + \mathrm{i}\int_a^b v(t)\,\mathrm{d}t.$$

Definition 3.1.2. Let $a, b \in \mathbf{R}$ and let $\gamma\colon [a, b] \to \mathbf{C}$ be a function.

(1) For each $k \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$, we say that γ is a *piecewise* $\mathcal{C}^k$*-curve* if there exist finitely many points $a = a_0 < a_1 < \cdots < a_n = b$ such that γ restricts to a $\mathcal{C}^k$-function on the subinterval $[a_{r-1}, a_r] \subseteq [a, b]$ for each $1 \leq r \leq n$.

(2) We say that γ is a *contour* if it is a continuous, piecewise $\mathcal{C}^1$-function.

(3) We say that a contour γ is *closed* if $\gamma(a) = \gamma(b)$.

opposite contour

Definition 3.1.3. Let $a, b \in \mathbf{R}$, let $\gamma\colon [a, b] \to \mathbf{C}$ be a contour, let $X \subseteq \mathbf{C}$ be a subset containing $\gamma([a, b])$, and let $f\colon X \to \mathbf{C}$ be a function. We define the *contour integral* $\int_\gamma f\,\mathrm{d}z$ *of* f *along* γ by

$$\int_\gamma f\,\mathrm{d}z := \sum_{r=1}^{n} \int_{a_{r-1}}^{a_r} f(\gamma(t))\gamma'(t)\,\mathrm{d}t,$$

where $a = a_0 < a_1 < \cdots < a_n = b$ are chosen so that γ is continuously differentiable along $[a_{r-1}, a_r]$ for $1 \leq r \leq n$. If $\Gamma = \gamma([a, b])$ and the parametrization γ is clear from the context, then we sometimes write $\int_\Gamma f\,\mathrm{d}z$ rather than $\int_\gamma f\,\mathrm{d}z$.

Convention 3.1.4. With the notation and hypotheses of Definition 3.1.3, we often abusively suppress the boundary points $a_1, \dots, a_{n-1}$ of γ, simply writing $\int_\gamma f \, \mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}t$.

Definition 3.1.5. Let $a, b \in \mathbf{R}$ and let $\gamma \colon [a, b] \to \mathbf{C}$ be a contour. We define the *arc length* $\mathrm{L}(\gamma)$ *of* γ by

$$\mathrm{L}(\gamma) \coloneqq \int_a^b |\gamma'(t)| \, \mathrm{d}t.$$

Lemma 3.1.6. *Let $U \subseteq \mathbf{C}$ be an open subset, let $a, b \in \mathbf{R}$, let $\gamma \colon [a, b] \to U$ be a contour, let $f \colon U \to \mathbf{C}$ be a continuous function, and let $M \in \mathbf{R}$. If $|f(z)| \leq M$ for each $z \in \Gamma$, then*

$$\left| \int_\gamma f \, \mathrm{d}z \right| \leq M \cdot \mathrm{L}(\gamma),$$

where $\mathrm{L}(\gamma)$ is the arc length of γ.

Proof. By definition of the contour integral (Definition 3.1.3) and [add reference], we have

$$\left| \int_\gamma f \, \mathrm{d}z \right| = \left| \int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}t \right| \leq \int_a^b |f(\gamma(t))\gamma'(t)| \, \mathrm{d}t = \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)| \, \mathrm{d}t.$$

By [add reference], we also have

$$\int_a^b |f(\gamma(t))| \cdot |\gamma'(t)| \, \mathrm{d}t \leq M \int_a^b |\gamma'(t)| \, \mathrm{d}t = M \cdot \mathrm{L}(\gamma),$$

as required. □

3.2 Goursat's lemma and Cauchy's theorem

Definition 3.2.1. Let $a_r \in \mathbf{R}$ for each $r \in \mathbf{Z} \cap [0, 3]$, let $\gamma \colon [a_0, a_3] \to \mathbf{C}$ be a contour, and let $\Gamma \coloneqq \gamma([a_0, a_3])$. We say that γ is *triangular* if it satisfies the following conditions:

- for each $1 \leq r \leq 3$, $\Gamma_r \coloneqq \gamma([a_{r-1}, a_r])$ is a line segment with endpoints $\gamma(a_{r-1})$, $\gamma(a_r)$;
- the contour γ is closed, so that Γ forms a triangle; and
- γ traces the triangle Γ in the counterclockwise direction.

Remark 3.2.2. With the notation of Definition 3.2.1 and Convention 3.1.4, if γ is a triangular contour, then it follows immediately from the definition of a contour integral (Definition 3.1.3) that

$$\int_\Gamma f \, \mathrm{d}z = \int_{\Gamma_1} f \, \mathrm{d}z + \int_{\Gamma_2} f \, \mathrm{d}z + \int_{\Gamma_3} f \, \mathrm{d}z.$$

Lemma 3.2.3 (Goursat)**.** *Let $U \subseteq \mathbf{C}$ be an open subset, let $x \in U$, and let $f \colon U \to \mathbf{C}$ be a continuous function that is complex differentiable $U - \{x\}$. For each triangular contour γ whose image is contained in U, we have $\int_\gamma f \, \mathrm{d}z = 0$.*

Proof. Choose a triangular contour γ as in Definition 3.2.1. Let $\Gamma \coloneqq \gamma([a_0, a_3])$ and let T denote the closed, triangular region bounded by Γ. There are four cases to consider:

(1) $x \notin T$;
(2) $x \in \{\gamma(a_0), \gamma(a_1), \gamma(a_2)\}$, i.e., x is a vertex of the triangle Γ; and
(3) $x \in T - \{\gamma(a_0), \gamma(a_1), \gamma(a_2)\}$, i.e., x lies in T, but is not a vertex of the triangle.

Consider Case (1), i.e., assume that $x \notin T$. Let b_r denote the midpoint of the line segment joining $\gamma(a_{r-1})$ to $\gamma(a_r)$ for $1 \leq r \leq 3$. The line segments joining the points b_1, b_2, and b_3 divide T into four triangular regions, which we will denote by $T_{1,r}$ for $0 \leq r \leq 3$. We denote the triangle bounding $T_{1,r}$ by $\Gamma_{1,r}$ for $0 \leq r \leq 3$. If $\Gamma_{1,r}$ contains a vertex of Γ, then it shares an angle with Γ, and two of its sides are proportional to two sides of Γ, so $\Gamma_{1,r}$ is proportional to Γ. It follows that the perimeter of $\Gamma_{1,r}$ is half that of Γ for each $0 \leq r \leq 3$. It also follows that each of the triangles $\Gamma_{1,r}$, $0 \leq r \leq 3$, is proportional to Γ with ration $1/2$. In particular, the parametrization γ of Γ induces a parametrization of $\Gamma_{1,r}$ for $0 \leq r \leq 3$ such that the arc length of the parametrization of $\Gamma_{1,r}$ is half that of Γ. It will be convenient, however, to parametrize each $\Gamma_{1,r}$ with a counterclockwise orientation, so for the "interior" triangle, we will be working rather with a parametrization corresponding to the contour opposite to γ. With these conventions, we let $\gamma_{1,r}$ denote the resulting parametrization of $\Gamma_{1,r}$ for $0 \leq r \leq 3$.

By [add reference], we have

$$\int_\gamma f \,\mathrm{d}z = \sum_{r=0}^{3} \int_{\gamma_{1,r}} f \mathrm{d}z,$$

where we parametrize each $\Gamma_{1,r}$ with a counterclockwise orientation. Repeating this process, i.e., subdividing each $\Gamma_{1,r}$ into four triangles $T_{2,r}$ for $0 \leq r \leq 3$, we find by induction that

$$\int_\gamma f \,\mathrm{d}z = \sum_{r=0}^{4^n-1} \int_{\gamma_{n,r}} f \,\mathrm{d}z$$

for each $n \in \mathbf{Z}_{>0}$.

We have

$$\sum_{r=0}^{3} \left| \int_{\gamma_{1,r}} f \,\mathrm{d}z \right| \geq \left| \sum_{r=0}^{3} \int_{\gamma_{1,r}} f \mathrm{d}z \right| = \left| \int_\gamma f \,\mathrm{d}z \right| =: I \geq 0.$$

Re-indexing if necessary, we may assume that

$$\left| \int_{\gamma_{1,0}} f \,\mathrm{d}z \right| \geq 2^{-2} I = 4^{-1} I.$$

Repeating this process, we may assume that

$$\left| \int_{\gamma_{n,0}} f \,\mathrm{d}z \right| \geq 4^{-n} I \tag{3.2.3.a}$$

for each $n \in \mathbf{Z}_{>0}$. For each such n, the arc length of $\gamma_{n,0}$ is $2^{-n}\,\mathrm{L}(\gamma)$, where $\mathrm{L}(\gamma)$ is the arc length of Γ, and the perimeter $P(\Gamma_{n,0})$ of $\Gamma_{n,0}$ is $2^{-n}\,\mathrm{P}(\Gamma)$, where $\mathrm{P}(\Gamma)$ is the perimeter of Γ.

For each $n \in \mathbf{Z}_{>0}$, let $T_{n,0}$ denote the triangular region bounded by $\Gamma_{n,0}$. The intersection $\bigcap_{n>0} T_{n,0}$ consists of a single point w: no two points w and w' are contained in a region bounded by a triangle of perimeter $< |w - w'|$; on the other hand, this intersection is nonempty by Cantor's intersection theorem (Theorem 1.3.36).

By hypothesis, f is complex differentiable at w. Let $\varepsilon \in \mathbf{R}_{>0}$. There exists $\delta \in \mathbf{R}_{>0}$ such that

$$\left| \frac{f(z) - f(w)}{z - w} - f'(w) \right| < \varepsilon \tag{3.2.3.b}$$

for each $z \in \mathbf{D}^*(w, \delta)$. Let $N \in \mathbf{Z}_{>0}$ such that $T_{N,0}$ is contained in $\mathbf{D}(w, \delta)$. As $\mathrm{P}(\Gamma_{N,0})$ is $2^{-N}\,\mathrm{P}(\Gamma)$, we have

$$T_{n,0} \subseteq \bar{\mathbf{D}}(w, 2^{-N}\,\mathrm{P}(\gamma)). \tag{3.2.3.c}$$

By the evident additivity of contour integrals, we have

$$\begin{aligned}
\left| \int_{\gamma_{N,0}} f \,\mathrm{d}z \right| &= \left| \int_{\gamma_{N,0}} (f - (f(w) + f'(w)(z-w)))\,\mathrm{d}z + \int_{\gamma_{N,0}} (f(w) + f'(w)(z-w))\,\mathrm{d}z \right| \\
&= \left| \int_{\gamma_{N,0}} (f - (f(w) + f'(w)(z-w)))\,\mathrm{d}z \right|.
\end{aligned}$$

By Lemma 3.1.6, (3.2.3.b), and (3.2.3.c), we therefore have

$$\left|\int_{\gamma_{N,0}} f\,\mathrm{d}z\right| = \left|\int_{\gamma_{N,0}} (f - (f(w) + f'(w)(z-w)))\,\mathrm{d}z\right| \leq \varepsilon\,\mathrm{P}(\gamma)\,\mathrm{L}(\gamma).$$

Combining this with (3.2.3.a), we find that

$$\left|\int_{\gamma} f\,\mathrm{d}z\right| \leq 4^N \left|\int_{\gamma_{N,0}} f\,\mathrm{d}z\right| \leq 4^N \varepsilon (2^{-N}\,\mathrm{P}(\Gamma))(2^{-N}\,\mathrm{L}(\gamma)) = \varepsilon\,\mathrm{P}(\Gamma)\,\mathrm{L}(\gamma).$$

As this is valid for each $\varepsilon \in \mathbf{R}_{>0}$, which completes the proof in Case (1).

Consider Case (2), i.e., suppose that x is a vertex of Γ. Without loss of generality, we may assume that $x = \gamma(a_0)$. Choose points $b_0 \in [\gamma(a_0), \gamma(a_1)]$ and $b_1 \in [\gamma(a_0), \gamma(a_2)]$ and subdivide T into triangular subregions by drawing each possible line segment with endpoints in $\{b_0, b_1, \gamma(a_1), \gamma(a_2)\}$. Choosing contours parametrizing the resulting triangles in T as in the previous case, it follows from Case (1) and Remark 3.2.2 that the only contribution to $\int_\gamma f\,\mathrm{d}z$ comes from the contour integral associated with the triangle with vertices x, b_0, and b_1. This triangle again satisfies the hypotheses of Case (2), but we can force the arc length of the corresponding contour to be arbitrarily small by choosing b_0 and b_1 sufficiently close to x.

Consider Case (3), i.e., suppose that $x \in T$ and that x is not a vertex. Subdivide T into triangular subregions by drawing the line segments connecting x to each of the vertices of Γ. The claim again follows from Case (2): x is a vertex of each of the new triangles. □

Proposition 3.2.4. *Let $U \subseteq \mathbf{C}$ be an open subset, let $a, b \in \mathbf{R}$, let $\gamma\colon [a,b] \to U$ be a closed contour, and let $f\colon U \to \mathbf{C}$ be a complex-differentiable function. If f' is continuous on U, then $\int_\gamma f'\,\mathrm{d}z = 0$.*

Proof. By the fundamental theorem of calculus and the hypothesis that $\gamma(a) = \gamma(b)$, we have

$$\int_\gamma f'\,\mathrm{d}z = \int_a^b f'(\gamma(t))\gamma'(t)\,\mathrm{d}t = f(\gamma(b)) - f(\gamma(a)) = 0,$$

as required. □

Theorem 3.2.5 (Cauchy)**.** *Let $U \subseteq \mathbf{C}$ be a convex open set, let $x \in U$, and let $f\colon U \to \mathbf{C}$ be a continuous function that is complex differentiable on $U - \{x\}$. The following properties hold:*

(1) *there exists a complex-differentiable function $F\colon U \to \mathbf{C}$ such that $f = F'$; and*
(2) *for each closed contour γ whose image is contained in U, $\int_\gamma f\,\mathrm{d}z = 0$.*

Proof. Consider Claim (1). Convexity of U means that, for each pair of distinct points $z, w \in U$, the line segment joining them is contained in U. Letting $\ell_{z,w} := |z - w|$, choose a contour $\gamma_{z,w}\colon [0, \ell_{z,w}] \to U$ parametrizing this line segment, and do so in such a way that:

- $\gamma_{z,w}$ is opposite to $\gamma_{w,z}$; and
- $\gamma'_{z,w}(t) = 1$ for each $t \in (0,1)$.

The hypothesis regarding $\gamma'_{z,w}$ implies that $\gamma_{z,w}$ is of arc length $\mathrm{L}(\gamma_{z,w}) = \ell_{z,w}$.

Fix a point $a \in U$. We define a function $F\colon U \to \mathbf{C}$ by

$$F(z) := \int_{\gamma_{a,z}} f\,\mathrm{d}\zeta.$$

We claim that F is complex differentiable on U and that its derivative is given by f.

For each pair of points $z, w \in U$, the triangular region T with vertices a, z, and w is contained in the convex set U. This is true even in the degenerate cases in which two of the vertices are equal and the triangle is only a line segment. Let Γ denote the boundary of T. We may assume without loss of generality that a, z, and w are arranged so that tracing Γ in the path $a \to z \to w \to a$ has a counterclockwise orientation. Parametrize Γ by a closed contour γ obtained from $\gamma_{a,z}$, $\gamma_{z,w}$, and $\gamma_{w,a}$.

We must show that, for each $\varepsilon \in \mathbf{R}_{>0}$, there exists $\delta \in \mathbf{R}_{>0}$ such that, for each $z \in \mathbf{D}^*(w, \delta)$,

$$\frac{F(z) - F(w)}{z - w} - f(w) \in \mathbf{D}(0, \varepsilon). \tag{3.2.5.a}$$

We first observe that

$$F(z) - F(w) = \int_{\gamma_{w,z}} f \,\mathrm{d}\zeta. \tag{3.2.5.b}$$

Indeed, we have

$$\begin{aligned}
0 &= \int_\gamma f \,\mathrm{d}\zeta && \text{Lemma 3.2.3}\\
&= \int_{\gamma_{a,z}} f \,\mathrm{d}\zeta + \int_{\gamma_{z,w}} f \,\mathrm{d}\zeta + \int_{\gamma_{w,a}} f \,\mathrm{d}\zeta && \text{Remark 3.2.2}\\
&= \int_{\gamma_{a,z}} f \,\mathrm{d}\zeta - \int_{\gamma_{w,z}} f \,\mathrm{d}\zeta - \int_{\gamma_{a,w}} f \,\mathrm{d}\zeta && \text{[add reference]}\\
&= F(z) - \int_{\gamma_{w,z}} f \,\mathrm{d}\zeta - F(w).
\end{aligned}$$

For each $z \neq w$, observe that

$$1 = \frac{z-w}{z-w} = \frac{\gamma_{w,z}(\ell_{w,z}) - \gamma_{w,z}(0)}{z-w} = \frac{1}{z-w} \int_0^{\ell_{w,z}} \gamma'(t) \,\mathrm{d}t = \frac{1}{z-w} \int_{\gamma_{w,z}} \mathrm{d}\zeta. \tag{3.2.5.c}$$

Combining (3.2.5.b) and (3.2.5.c),

$$\frac{F(z) - F(w)}{z - w} - f(w) = \frac{1}{z-w} \int_{\gamma_{w,z}} f \,\mathrm{d}\zeta - f(w) = \frac{1}{z-w} \int_{\gamma_{w,z}} (f - f(w)) \,\mathrm{d}\zeta$$

for each $z \neq w$.

Returning now to (3.2.5.a), for each $\varepsilon \in \mathbf{R}_{>0}$, continuity of f implies that there exists $\delta \in \mathbf{R}_{>0}$ such that $f(\zeta) - f(w) \in \mathbf{D}(0, \varepsilon)$ for each $\zeta \in \mathbf{D}(w, \delta)$. In particular, if $z \in \mathbf{D}(w, \delta)$, then the image of $\gamma_{w,z}$ is contained in $\mathbf{D}(w, \delta)$ and, hence, $f(\gamma_{w,z}(t)) - f(w) \in \mathbf{D}(0, \varepsilon)$ for each $t \in [0, \ell_{w,z}]$. Thus, for each $z \in \mathbf{D}(w, \delta)$, we have

$$\left| \frac{1}{z-w} \int_{\gamma_{w,z}} (f - f(w)) \,\mathrm{d}\zeta \right| = \frac{1}{|z-w|} \left| \int_{\gamma_{w,z}} (f - f(w)) \,\mathrm{d}\zeta \right| < \frac{\varepsilon \,\mathrm{L}(\gamma_{w,z})}{\ell_{w,z}} = \varepsilon,$$

which completes the proof of Claim (1).

Claim (2) follows from Proposition 3.2.4 and (1). □

3.3 Analyticity of complex-differentiable functions

Recall from Definition 1.3.11 that a connected component of a topological space X is a connected subspace that is maximal with respect to inclusion.

Lemma 3.3.1. *If $\Gamma \subseteq \mathbf{C}$ is the image of a closed contour, then:*

(1) *Γ is a quasi-compact and, hence, closed and bounded subset of $\mathbf{C}$; and*

(2) *$\mathbf{C} - \Gamma$ has exactly one unbounded connected component.*

Proof. By Proposition 1.3.28, the image of a quasi-compact space under a continuous map is quasi-compact. By the Heine-Borel theorem (Theorem 1.3.33), each quasi-compact subspace of $\mathbf{C}$ is closed and bounded.

Since Γ is bounded, there exists $r \in \mathbf{R}_{>0}$ such that $\Gamma \subseteq \mathbf{D}(0, r)$. By Example 1.3.7, the complement $\mathbf{C} - \mathbf{D}(0, r)$ is connected, so it must be contained in a single connected component of $\mathbf{C} - \mathbf{D}(0, r)$. Each other connected component is contained in $\mathbf{D}(0, r)$ and is therefore bounded. $\square$

Lemma 3.3.2. *Let $U \subseteq \mathbf{C}$ be an open subset, let $a, b \in \mathbf{R}$, let $\gamma\colon [a, b] \to U$ be a contour, let $\Gamma \coloneqq \gamma([a, b])$, and let $g\colon U \to \mathbf{C}$ be a continuous function. The function $\varphi\colon U - \Gamma \to \mathbf{C}$ given by*

$$\varphi(z) \coloneqq \int_\gamma \frac{g}{w - z}\,\mathrm{d}w$$

is analytic and, in particular, continuous.

Proof. Let $\zeta \in U - \Gamma$. By Lemma 3.3.1.(1), $U - \Gamma$ is open in $\mathbf{C}$. Choose $r \in \mathbf{R}_{>0}$ such that $\mathbf{D}(\zeta, r) \subseteq U - \Gamma$. As we can express $U - \Gamma$ as the union of subsets of the form $\mathbf{D}(\zeta, r)$, it suffices to show that the restriction of φ to $\mathbf{D}(\zeta, r)$ admits a power-series representation.

Fix $z \in \mathbf{D}(\zeta, r)$ and $t \in [a, b]$. We have

$$\frac{1}{\gamma(t) - z} = \frac{1}{(\gamma(t) - \zeta) - (z - \zeta)} = \frac{1}{\gamma(t) - \zeta} \cdot \frac{1}{1 - \frac{z - \zeta}{\gamma(t) - \zeta}}.$$

Since $\mathbf{D}(\zeta, r) \subseteq U - \Gamma$, we have $|\gamma(t) - \zeta| \geq r$, whence

$$\left|\frac{z - \zeta}{\gamma(t) - \zeta}\right| \leq \frac{|z - \zeta|}{r} < 1.$$

Letting $u(t) = (z - \zeta)/(\gamma(t) - \zeta)$, the geometric series $\sum_{n \geq 0} u(t)^n$ therefore converges uniformly on $[a, b]$, and

$$\frac{1}{\gamma(t) - z} = \frac{1}{\gamma(t) - \zeta} \sum_{n \geq 0} u(t)^n = \frac{1}{\gamma(t) - \zeta} \sum_{n \geq 0} \left(\frac{z - \zeta}{\gamma(t) - \zeta}\right)^n = \sum_{n \geq 0} \frac{(z - \zeta)^n}{(\gamma(t) - \zeta)^{n+1}}.$$

Taking notational liberties with the boundary points $a_1, \ldots, a_{n-1}$ of γ, it follows that

$$\varphi(z) = \int_\gamma \frac{g}{w - z}\,\mathrm{d}w = \int_a^b \frac{g(\gamma(t))\gamma'(t)}{\gamma(t) - z}\,\mathrm{d}t = \int_a^b \sum_{n \geq 0} \frac{(z - \zeta)^n g(\gamma(t))\gamma'(t)}{(\gamma(t) - \zeta)^{n+1}}\,\mathrm{d}t.$$

By uniform convergence, we may rewrite this last term as

$$\sum_{n \geq 0} \int_a^b \frac{(z - \zeta)^n g(\gamma(t))\gamma'(t)}{(\gamma(t) - \zeta)^{n+1}}\,\mathrm{d}t = \sum_{n \geq 0} \left(\int_a^b \frac{g(\gamma(t))\gamma'(t)}{(\gamma(t) - \zeta)^{n+1}}\,\mathrm{d}t\right)(z - \zeta)^n.$$

The integrand in the nth term of this last series is independent of z; the modulus of the numerator is bounded on each piece of γ as it is continuous and the image of a quasi-compact space under a continuous map into $\mathbf{C} \simeq \mathbf{R}^2$ is quasi-compact Proposition 1.3.28 and therefore, by the Heine-Borel theorem (Theorem 1.3.33) bounded; and the modulus of the denominator is $\geq r^{n+1}$. The coefficients of this series are therefore well-defined, and we have the desired power-series representation of φ on $\mathbf{D}(\zeta, r)$. $\square$

Definition 3.3.3. Let $a, b \in \mathbf{R}$, let $\gamma\colon [a, b] \to \mathbf{C}$ be a closed contour, and let $\zeta \in \mathbf{C} - \gamma([a, b])$. We define the *winding number* $\operatorname{wind}(\gamma, \zeta)$ *of* γ *around* ζ by

$$\operatorname{wind}(\gamma, \zeta) \coloneqq \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\mathrm{d}z}{z - \zeta}.$$

Example 3.3.4. Let $a \in \mathbf{C}$, let $r \in \mathbf{R}_{>0}$, and let $\gamma\colon [0, 2\pi] \to \mathbf{C}$ denote the closed contour given by

$$\gamma(t) := a + r\exp(\mathrm{i}t),$$

which traverses the circle $\partial\mathbf{D}(a, r)$ of radius r centered at a in the counterclockwise direction. We have $\gamma'(t) = \mathrm{i}r\exp(\mathrm{i}t)$, whence

$$\operatorname{wind}(\gamma, a) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\mathrm{d}w}{w-a} = \frac{1}{2\pi\mathrm{i}}\int_0^{2\pi} \frac{\mathrm{i}r\exp(\mathrm{i}t)}{(a + r\exp(\mathrm{i}t)) - a}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}t = 1.$$

Proposition 3.3.5. *Let $a, b \in \mathbf{R}$, let $\gamma\colon [a, b] \to \mathbf{C}$ be a closed contour, and let $\Gamma := \gamma([a, b])$.*

(1) *The function $f\colon z \mapsto \operatorname{wind}(\gamma, z)\colon \mathbf{C} - \Gamma \to \mathbf{C}$ is analytic and, in particular, continuous.*
(2) *For each $z \in \mathbf{C} - \Gamma$, the winding number $\operatorname{wind}(\gamma, z)$ is an integer.*
(3) *The function f is constant on each connected component of $\mathbf{C} - \Gamma$.*

Proof. Choose $a = a_0 < a_1 < \cdots < a_n = b$ such that γ is a $\mathcal{C}^1$-function on $[a_{r-1}, a_r]$ for each $1 \le r \le n$.

Claim (1) is the special case of Lemma 3.3.2 in which $U = \mathbf{C}$, $\varphi = f$, and g denotes the constant function with value $(2\pi\mathrm{i})^{-1}$.

Consider Claim (2). Let $z \in \mathbf{C} - \Gamma$. The function $\psi\colon [a, b] \to \mathbf{C}$ defined by

$$\psi(s) := \int_a^s \frac{\gamma'(t)}{\gamma(t) - z}\,\mathrm{d}t$$

is continuous: the denominator of the integrand $\gamma - z$ is nonvanishing; and the numerator γ' is piecewise continuous. The function $\varphi := \exp\circ\psi\colon [a, b] \to \mathbf{C}$ is therefore also continuous. Differentiating with respect to t, we find that

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{\gamma'(t)}{\gamma(t) - z} \tag{3.3.5.a}$$

for each $t \in [a, b] - \{a_0, \ldots, a_n\}$. The function $\varphi/(\gamma - z)$ is also continuous, and, on $[a, b] - \{a_0, \ldots, a_n\}$, its first derivative is

$$\frac{(\gamma - z)\varphi' - \varphi\gamma'}{(\gamma - z)^2},$$

which is identically zero by (3.3.5.a). It follows that $\varphi/(\gamma - z)$ is differentiable on (a, b): the left and right derivatives at $a_1, \ldots, a_{n-1}$ are all zero. It follows that $\varphi/(\gamma - z)$ is constant on $[a, b]$. As $\gamma(a) = \gamma(b)$ and $\varphi(a) = 1$, we have

$$1 = \frac{\varphi(b)}{\gamma(b) - z}\cdot\frac{\gamma(a) - z}{\varphi(a)} = \frac{\varphi(b)}{\varphi(a)} = \varphi(b) = \exp(2\pi\mathrm{i}\operatorname{wind}(\gamma, z)).$$

It follows that $\operatorname{wind}(\gamma, z)$ is an integer.

Consider Claim (3). By (1) and (2), we have a continuous function $f\colon \mathbf{C} - \Gamma \to \mathbf{Z}$, where $\mathbf{Z}$ is equipped with the subspace topology. Its image is connected by continuity (Proposition 1.3.5), and the only connected subsets of $\mathbf{Z}$ are the singletons. $\square$

Example 3.3.6. With the notation and hypotheses of Example 3.3.4, that example and Proposition 3.3.5.(3) together imply that $\operatorname{wind}(\gamma, z) = 1$ for each $z \in \mathbf{D}(a, r)$.

Proposition 3.3.7 (Cauchy formula)**.** *Let $U \subseteq \mathbf{C}$ be a convex open subset, let γ be a closed contour whose image Γ is contained in U, and let $f\colon U \to \mathbf{C}$ be a complex-differentiable function. For each $z \in U - \Gamma$, we have*

$$f(z)\operatorname{wind}(\gamma, z) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{f}{w - z}\,\mathrm{d}w. \tag{3.3.7.a}$$

Proof. Let $z \in U - \Gamma$. Consider the function $g\colon U \to \mathbf{C}$ defined by

$$g(w) := \frac{f(w) - f(z)}{w - z}$$

for $w \neq z$, and by $g(z) := f'(z)$. The hypothesis that f is complex differentiable implies that g is continuous on U and complex differentiable on $U - \{z\}$. By Theorem 3.2.5, we deduce that

$$0 = \int_\gamma g \,\mathrm{d}w = \int_\gamma \frac{f(w) - f(z)}{w - z} \,\mathrm{d}w = \int_\gamma \frac{f}{w - z} \,\mathrm{d}w - \int_\gamma \frac{f(z)}{w - z} \,\mathrm{d}w.$$

Rearranging and scaling by $(2\pi \mathrm{i})^{-1}$, it now follows from the definition of the winding number (Definition 3.3.3) that

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f}{w - z} \,\mathrm{d}w = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(z)}{w - z} \,\mathrm{d}w = \frac{f(z)}{2\pi\mathrm{i}} \int_\gamma \frac{\mathrm{d}w}{w - z} = f(z) \operatorname{wind}(\gamma, z),$$

as required. □

Theorem 3.3.8. *Let $U \subseteq \mathbf{C}$ be an open subset. A function $f\colon U \to \mathbf{C}$ is complex differentiable if and only if f is analytic.*

Proof. Suppose that f is complex differentiable. Let $a \in U$, choose $r \in \mathbf{R}_{>0}$ such that $\mathbf{D}(a, r) \subseteq U$, and let $z \in \mathbf{D}(a, r)$. Let $\gamma\colon [0, 2\pi] \to U$ be the closed contour given by $\gamma(t) := a + r \exp(\mathrm{i}t)$. By Example 3.3.6, we have $\operatorname{wind}(\gamma, z) = 1$. The subset $\mathbf{D}(a, r)$ is convex, so Proposition 3.3.7 implies that

$$f(z) = f(z) \operatorname{wind}(\gamma, z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f}{w - z} \,\mathrm{d}w.$$

By Lemma 3.3.2, the right-hand side of this equation is an analytic function of z on $\mathbf{D}(a, r)$.

The converse was already established in Proposition 2.4.4. □

References