Topology of algebraic varieties

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Contents

2 Analytic functions

Motivation. In this chapter, we turn our attention to complex-analytic functions. For the purposes of this course, the fundamental properties of complex-analytic functions are the following:

(1) Complex-analytic functions are holomorphic, i.e., complex differentiable.

(2) Complex-analytic functions are smooth functions when regarded as real-valued functions.

(3) Complex-analytic functions satisfy the Implicit and Inverse Function Theorems.

(4) Local complex-analytic isomorphisms are orientation-preserving maps: when regarded as real-valued maps, their Jacobian matrices define $\mathbf{C}$-linear morphisms.

2.1 Formal power series and analytic functions

Much of this section follows the exposition of [Lan99, Chapter II].

Formal power series

Definition 2.1.1. Let $a = (a_1, \dots, a_n) \in \mathbf{C}^n$ and let $t = (t_1, \dots, t_n)$ be indeterminates. A *formal power series in t centered at a* is an expression of the form

$$f(t_1, \dots, t_n) = \sum_{(r_1,\dots,r_n)\in\mathbf{Z}^n_{\geq 0}} c_{r_1,\dots,r_n}(t_1 - a_1)^{r_1} \cdots (t_n - a_n)^{r_n}, \tag{2.1.1.a}$$

where $c_{r_1,\cdots,r_n} \in \mathbf{C}$ for each $(r_1, \dots, r_n) \in \mathbf{Z}^n_{\geq 0}$. We refer to the $c_{r_1,\dots,r_n}$ as the *coefficients* of the power series.

Remark 2.1.2. The qualifier "formal" in the expression "formal power series" indicates that the series is not required to converge in any sense: it has a purely symbolic meaning.

Notation 2.1.3. We denote by $\mathbf{C}[\![z_1, \dots, z_n]\!]$ the set of formal power series in $z_1, \dots, z_n$ centered at 0.

Remark 2.1.4. To be precise, when we speak of the convergence of the series $f(t_1, \ldots, t_n)$ of (2.1.1.a) for some value $(t_1, \ldots, t_n) \in \mathbf{C}^n$, we shall mean that the *total-degree partial sums*

$$s_k \coloneqq \sum_{|r_1,\ldots,r_n| \leq k} c_{r_1,\ldots,r_n}(t_1 - a_1)^{r_1} \cdots (t_n - a_n)^{r_n}$$

form a convergent sequence $(s_k)_{k \in \mathbf{Z}_{\geq 0}}$, where $|r_1, \ldots, r_n| \coloneqq r_1 + \cdots r_n$ is the *total degree*. For our purposes, we will consider in practice only absolutely convergent series, and we could therefore replace the total-degree-partial-sum sequence by the sequence associated with any other ordering of the summands of the series (2.1.1.a).

Definition 2.1.5. Let $X \subseteq \mathbf{C}^n$.

(1) We say that the formal power series (2.1.1.a) *converges absolutely on* X if, for each $(z_1, \ldots, z_n) \in X$, the series

(2.1.5.a)
$$\sum_{(r_1,\ldots,r_n) \in \mathbf{Z}^n_{\geq 0}} |c_{r_1,\ldots,r_n}(z_1 - a_1)^{r_1} \cdots (z_n - a_n)^{r_n}|$$

converges.

(2) We say that the formal power series (2.1.1.a) *converges uniformly absolutely on* X if the series (2.1.5.a) converges uniformly on X. Recall that a sequence of functions $(f_n \colon X \to \mathbf{C})_{n \in \mathbf{Z}_{\geq 0}}$ converges uniformly to $f \colon X \to \mathbf{C}$ if, for each $\varepsilon \in \mathbf{R}_{>0}$, there exists $N \in \mathbf{Z}_{\geq 0}$ such that, for each $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

Notation 2.1.6. We denote by $\mathbf{C}\{z_1, \ldots, z_n\} \subseteq \mathbf{C}[\![z_1, \ldots, z_n]\!]$ the subset consisting of formal power series in $z_1, \ldots, z_n$ centered at 0 that converge absolutely in a neighborhood of 0.

Example 2.1.7. Let $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$. The *geometric series with ratio* z is the formal power series

(2.1.7.a)
$$\sum_{(r_1,\ldots,r_n) \in \mathbf{Z}^n_{\geq 0}} z_1^{r_1} \cdots z_n^{r_n}$$

We claim that this series is uniformly absolutely convergent on each quasi-quasi-compact subset $K \subseteq \mathbf{D}(0, 1)^n$. There exists $0 \leq r < 1$ such that $K \subseteq \bar{\mathbf{D}}(0, r)^n$, and $\bar{\mathbf{D}}(0, r)^n$ is quasi-compact by Theorem 1.3.33, so we may assume without loss of generality that $K = \bar{\mathbf{D}}(0, r)^n$.

For each $z = (z_1, \ldots, z_n) \in \bar{\mathbf{D}}(o, r)^n$, if the iterated series

$$\sum_{r_n} \cdots \sum_{r_1} z_1^{r_1} \cdots z_n^{r_n} = \sum_{r_n} \left(\sum_{r_{n-1}} \cdots \sum_{r_1} z_1^{r_1} \cdots z_{n-1}^{r_{n-1}} \right) z_n^{r_n}$$

converges absolutely on K, then (2.1.7.a) converges absolutely to the same value. By induction on n, the iterated series is given by

$$\begin{aligned}\sum_{r_n} \cdots \sum_{r_2} \left(\sum_{r_1} z_1^{r_1} \right) z_2^{r_2} \cdots z_{n-1}^{r_{n-1}} z_n^{r_n} &= \left(\sum_{r_1} z_1^{r_1} \right) \left(\sum_{r_n} \cdots \sum_{r_2} z_2^{r_2} \cdots z_{n-1}^{r_{n-1}} z_n^{r_n} \right) \\ &= \left(\sum_{r_1} z_1^{r_1} \right) \left(\sum_{r_2} z_2^{r_2} \right) \cdots \left(\sum_{r_n} z_n^{r_n} \right)\end{aligned}$$

on K. By the well-known case in which $n = 1$, this is a product of finitely many series converging uniformly absolutely on K, so, by [add reference], it converges uniformly absolutely on K to the value

$$\frac{1}{(1 - z_1) \cdots (1 - z_n)}.$$

Proposition 2.1.8. *Let $a \in \mathbf{C}$, let $c_n \in \mathbf{C}$ for each $n \in \mathbf{Z}_{\geq 0}$, let $f(z) \coloneqq \sum_{n \in \mathbf{Z}_{\geq 0}} c_n(z - a)^n$ be the associated formal power series, and let $L \coloneqq \limsup_{n \to \infty} |c_n|^{\frac{1}{n}}$.*

(1) *If* $L = \infty$*, then* $f(z)$ *converges absolutely uniformly at* $z = a$ *and diverges at each other point* $z \in \mathbf{C}$*.*

(2) *If* $0 < L < \infty$*, then* $f(z)$ *converges uniformly absolutely on* $\mathbf{D}(a, r)$*, where* $r \coloneqq L^{-1}$*.*

(3) *If* $L = 0$*, then* $f(z)$ *converges uniformly absolutely on* $\mathbf{C}$*.*

Proof. See [Lan99, Chapter II, Theorem 2.6]. $\square$

Remark 2.1.9. We will establish a less precise but more general result in the several-variable case Lemma 2.1.10.

Lemma 2.1.10 (Abel)**.** *Consider a formal power series of the form* (2.1.5.a)*. Let* $(w_1, \dots, w_n) \in \mathbf{C}^n$*, let* $M \in \mathbf{R}_{>0}$*, and suppose that*

$$|c_{r_1,\dots,r_n}(w_1 - a_1)^{r_1} \cdots (w_n - a_n)^{r_n}| < M$$

for each $(r_1, \dots, r_n) \in \mathbf{Z}^n_{\geq 0}$*. The series* $f(z_1, \dots, z_n)$ *converges uniformly absolutely on each quasi-compact subset* K *of the polydisk* $D \coloneqq \mathbf{D}(a_1, \rho_1) \times \cdots \times \mathbf{D}(a_n, \rho_n)$*, where* $\rho_k \coloneqq |w_k - a_k|$ *for each* $1 \leq k \leq n$*.*

Proof. Let K be a quasi-compact subset of D. We may assume without loss of generality that $\rho_k > 0$ for each $1 \leq k \leq n$, or else $D = \varnothing$ and the claim is vacuous. As $K \subseteq D$, we have

$$\delta_k \coloneqq \sup_{(z_1,\dots,z_n)\in K} \frac{|z_k - a_k|}{\rho_k} < 1 \tag{2.1.10.a}$$

for each $1 \leq k \leq n$. Thus, for each $(z_1, \dots, z_n) \in K$ and each $(r_1, \dots, r_n) \in \mathbf{Z}^n_{\geq 0}$, we have

$$|c_{r_1,\dots,r_n}(z_1 - a_1)^{r_1} \cdots (z_n - a_n)^{r_n}| \leq |c_{r_1,\dots,r_n}|\rho_1^{r_1} \cdots \rho_n^{r_n}\delta_1^{r_1} \cdots \delta_n^{r_n} \leq M\delta_1^{r_1} \cdots \delta_n^{r_n}$$

By (2.1.10.a), $\delta_k < 1$ for each $1 \leq k \leq n$, so the geometric series $\sum_{(r_1,\dots,r_n)} M\delta_1^{r_1} \cdots \delta_n^{r_n}$ converges absolutely as observed in Example 2.1.7. $\square$

Analytic functions

Definition 2.1.11. Let $U \subseteq \mathbf{C}^n$ be an open subset.

(1) Let $a = (a_1, \dots, a_n) \in U$. We say that the function $f \colon U \to \mathbf{C}$ is *analytic at* a if there exists an open neighborhood $a \in V \subseteq U$ and a power series expansion

$$f(z) = \sum_{(r_1,\dots,r_n)\in\mathbf{Z}^n_{\geq 0}} c_{r_1\cdots r_n}(z_1 - a_1)^{r_1} \cdots (z_n - a_n)^{r_n}$$

with complex coefficients converging uniformly absolutely in each quasi-compact subset of V.

(2) We say that the function $f \colon U \to \mathbf{C}$ is *analytic on* U if it is so at each $a \in U$.

(3) We say that the function $f = (f_1, \dots, f_m) \colon U \to \mathbf{C}^m$ is *analytic on* U if f_k is analytic on U for each $1 \leq k \leq m$.

Exercise 2.1.12. If $f \colon U \to \mathbf{C}$ is an analytic function on the open subset $U \subseteq \mathbf{C}^n$, then f is continuous. [Hint: Uniform convergence.]

Exercise 2.1.13. Let $U \subseteq \mathbf{C}^n$ and $V \subseteq \mathbf{C}^m$ be open subsets.

(1) If $f, g \colon U \to \mathbf{C}^m$ are analytic on U, then so is $f + g$.

(2) If $f, g \colon U \to \mathbf{C}^m$ are analytic on U, then so is $f \cdot g$.

(3) If $f \colon U \to V$ and $g \colon V \to \mathbf{C}^k$ are analytic on U and V, respectively, then $g \circ f \colon U \to \mathbf{C}^k$ is analytic on U.

2.2 Differentiability of real functions

Definition 2.2.1. Let $X \subseteq \mathbf{R}^n$, let $f\colon X \to \mathbf{R}^m$ be a function, and let $x \in X$.

(1) If X is open in $\mathbf{R}^n$, then we say that f is *differentiable at* x if there exists an $\mathbf{R}$-linear map $\mathrm{d}f_x\colon \mathbf{R}^n \to \mathbf{R}^m$ such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - \mathrm{d}f_x(h)}{|h|} = 0 \tag{2.2.1.a}$$

More precisely, this notation means that, for each $\varepsilon \in \mathbf{R}_{>0}$, there exists $\delta \in \mathbf{R}_{>0}$ such that

$$\frac{f(x+h) - f(x) - \mathrm{d}f_x(h)}{|h|} \in \mathbf{D}(0, \varepsilon)$$

for each $h \in \mathbf{R}^n - \{0\}$ such that $x + h \in \mathbf{D}(0, \delta)$.

(2) We say that f is *differentiable on* X if it is so at each $x \in U$.

(3) More generally, we say that f is *differentiable on* X if, for each $x \in X$, there exists an open subset $U \subseteq \mathbf{R}^n$ containing x and a differentiable function $F\colon U \to \mathbf{R}^m$ such that $F|_{X\cap U} = f|_{X\cap U}$.

(4) If f is differentiable at x, then we refer to the $\mathbf{R}$-linear map $\mathrm{d}f_x$ of (1) as the *differential of* f *at* x, and we refer to its matrix with respect to the standard bases as the *Jacobian matrix* $\mathrm{J}_f(x)$ *of* f *at* x.

Remark 2.2.2. If $X \subseteq \mathbf{R}^n$ and $f\colon X \to \mathbf{R}$ is differentiable at $x \in X$, then the Jacobian matrix $\mathrm{J}_f(x)$ is of the form

$$\mathrm{J}_f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} (x),$$

where $f_r\colon \mathbf{R}^n \to \mathbf{R}$ is the composite of f with the projection $\mathbf{R}^m \to \mathbf{R}$ onto the rth coordinate for $1 \leq r \leq m$.

Definition 2.2.3. Let $X \subseteq \mathbf{R}^n$, let $f = (f_1, \dots, f_m)\colon X \to \mathbf{R}^m$ be a function, and let $k \in \mathbf{Z}_{\geq 0}$.

(1) If X is open in $\mathbf{R}^n$, then we say that f is *of continuity class* $\mathcal{C}^k$ or that f is a $\mathcal{C}^k$*-function* if, for each integer $1 \leq s \leq m$, for each integer $0 \leq r \leq k$, and for each partition $r = r_1 + \cdots + r_n$, the partial derivative

$$\frac{\partial^r f_s}{(\partial x_1)^{r_1} \cdots (\partial x_n)^{r_n}}$$

exists and is continuous on X. By convention, if $k = 0$, this condition means precisely that f_s is continuous for each $1 \leq s \leq m$.

(2) For an arbitrary subspace $X \subseteq \mathbf{R}^n$, we say that f is *of continuity class* $\mathcal{C}^k$ or that f is a $\mathcal{C}^k$*-function* if, for each $x \in X$, there exist an open subset $U \subseteq \mathbf{R}^n$ containing x and a $\mathcal{C}^k$-function $F\colon U \to \mathbf{R}$ such that $F|_{X\cap U} = f|_{X\cap U}$.

(3) We say that f is *smooth* or that f is a $\mathcal{C}^\infty$*-function* if it is a $\mathcal{C}^k$-function for each $k \geq 0$.

Proposition 2.2.4. *If $X \subseteq \mathbf{R}^n$ and $f\colon X \to \mathbf{R}^m$ is a $\mathcal{C}^1$-function, then f is differentiable in the sense of Definition* 2.2.1.(3).

2.3 Differentiability of complex functions

Definition 2.3.1. The *standard bijection* $\mathbf{R}^{2n} \to \mathbf{C}^n$ is the bijection $\mathbf{R}^{2n} \rightsquigarrow \mathbf{C}^n$ given by

$$(x_1, y_1, \dots, x_n, y_n) \mapsto (x_1 + \mathrm{i}y_n, \dots, x_n + \mathrm{i}y_n).$$

We let $\nu_n\colon \mathbf{C}^n \to \mathbf{R}^{2n}$ denote the inverse bijection, and we also refer to it as the standard bijection.

Definition 2.3.2. Let $X \subseteq \mathbf{C}^n$, let $f\colon X \to \mathbf{C}^m$ be a function, and let $k \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$. We say that f is *of continuity class* $\mathcal{C}^k$ or that f is a $\mathcal{C}^k$*-function* if the composite

$$\nu_n(X) \xrightarrow{\nu_n^{-1}} X \xrightarrow{f} \mathbf{C}^m \xrightarrow{\nu_m} \mathbf{R}^{2m}$$

is a $\mathcal{C}^k$-function, where ν_n and ν_m are the standard bijections of Definition 2.3.1.

Definition 2.3.3. Let $X \subseteq \mathbf{C}^n$ be a subset, let $a = (a_1, \ldots, a_n) \in X$, and let $f\colon X \to \mathbf{C}$ be a function.

(1) Suppose that X is open and that $n = 1$. We say that f is *complex differentiable at a* if the limit

$$f'(a) \coloneqq \lim_{z \to a} \frac{f(z) - f(a)}{z - a} \tag{2.3.3.a}$$

exists. In other words, f is complex differentiable at a if there exists $f'(z) \in \mathbf{C}$ such that, for each $\varepsilon \in \mathbf{R}_{>0}$, there exists $\delta \in \mathbf{R}_{>0}$ such that, for each $z \in \mathbf{D}(a, \delta)$, we have

$$\frac{f(z) - f(a)}{z - a} \in \mathbf{D}(f'(a), \varepsilon).$$

(2) Suppose that X is open. Let $z_1, \ldots, z_n$ denote the standard coordinate functions on $\mathbf{C}^n$ and let $1 \leq r \leq n$. By Exercise 1.1.4, for some $\varepsilon_1, \ldots, \varepsilon_n \in \mathbf{R}_{>0}$, the polydisk $\mathbf{D}(a_1, \varepsilon_1) \times \cdots \times \mathbf{D}(a_n, \varepsilon_n)$ is contained in X. We say that f is *complex differentiable at z with respect to the coordinate z_k* if the composite

$$\mathbf{D}(a_k, \varepsilon_r) \xrightarrow{\iota_k} \mathbf{D}(a_1, \varepsilon_1) \times \cdots \times \mathbf{D}(a_n, \varepsilon_n) \hookrightarrow X \xrightarrow{f} \mathbf{C}$$

is complex differentiable at a_r, where $\iota_k(z) = (a_1, \ldots, a_{k-1}, z, a_{k+1}, \ldots, a_n)$. In other words, f is complex differentiable at a with respect to z_k if the single-variable function

$$z \mapsto f(a_1, \ldots, a_{k-1}, z, a_{k+1}, \ldots, a_n)\colon \mathbf{D}(a_k, \varepsilon_k) \to \mathbf{C}$$

is holomorphic at a_k.

(3) Suppose that X is open. We say that f is *complex differentiable at a* if it satisfies the following conditions:

(a) f is continuous at z; and

(b) f is complex differentiable at a with respect to each of the variables $z_1, \ldots, z_n$.

(4) We say that f is *complex differentiable* if it is so at each $a \in X$.

(5) If X is an arbitrary subset, then we say that f is *complex differentiable* if, for each $a \in X$, there exist an open subset $U \subseteq \mathbf{C}^n$ containing a and a complex differentiable function $F\colon U \to \mathbf{C}$ such that $F|_{X \cap U} = f|_{X \cap U}$.

Definition 2.3.4. Let $X \subseteq \mathbf{C}^n$. A function $f\colon X \to \mathbf{C}$ is *holomorphic* if it is complex differentiable on X.

Remark 2.3.5. As we will see in Proposition 2.4.4, each analytic function is complex differentiable. Conversely, as we will see in Theorem 3.3.8, each complex-differentiable function is analytic. One may therefore employ the terms "analytic", "complex differentiable", and "holomorphic" interchangeably.

Of the two implications, the former (Proposition 2.4.4) is rather formal, and it is also this one that will be of use to us. The converse implication (Theorem 3.3.8) is a much more profound, although the most interesting ingredients are contained in the single-variable version of the Cauchy integral formula. The generalization to several variables is straightforward. This implication will not play a determinative role in the sequel, but it is valuable cultural knowledge and we include a comprehensive discussion for the sake of completeness.

Remark 2.3.6. A deep theorem due to Hartogs ([Har06]) implies that Condition 2.3.3.(3).(b) implies Condition 2.3.3.(3).(a): each function $f: U \to \mathbf{C}$ that is complex differentiable separately with respect to each variable is necessarily continuous. We will not appeal to this result in the sequel.

Example 2.3.7. The functions $\mathrm{Re}: \mathbf{C} \to \mathbf{R}$ and $\mathrm{Im}: \mathbf{C} \to \mathbf{R}$ are of continuity class $\mathcal{C}^\infty$ but neither of them is complex differentiable: their partial derivatives in the directions of the real and imaginary axes exist and are continuous to all orders, but they are not equal:

$$\frac{\partial \operatorname{Re}}{\partial x} = 1 \neq 0 = \frac{\partial \operatorname{Re}}{\partial y} \qquad \text{and} \qquad \frac{\partial \operatorname{Im}}{\partial x} = 0 \neq -\mathrm{i} = \frac{\partial \operatorname{Im}}{\partial y},$$

where x is the real-axis coordinate function and y the imaginary-axis coordinate function.

2.4 Complex-differentiability of analytic functions

Proposition 2.4.1. *Let $U \subseteq \mathbf{C}$ be an open subset and let $f, g: U \to \mathbf{C}$ be complex differentiable functions.*

(1) *The sum $f+g$ is complex differentiable with derivative $f'+g'$.*

(2) *The product fg is complex differentiable with derivative $f'g+fg'$.*

(3) *If g does not vanish on U, then the quotient f/g is complex differentiable with derivative $(f'g-fg')/g^2$.*

Proof. The proof of each assertion is essentially identical to the proof of the analogous assertion for real-differentiable functions in real analysis, proceeding by manipulation of the relevant difference quotients (2.3.3.a). See [Lan99, Chapter I, §5] for the details. ☐

Proposition 2.4.2. *Let $U, V \subseteq \mathbf{C}$ be open subsets and let $f: U \to V$ and $g: V \to \mathbf{C}$ be complex-differentiable functions. The composite $g \circ f: U \to \mathbf{C}$ is complex differentiable with derivative $(g' \circ f)f'$.*

Proof. The proof is once again essentially identical to the proof of the analogous assertion for real-differentiable functions, proceeding by manipulation of the relevant difference quotients (2.3.3.a). See [Lan99, Chapter I, §5] for the details. ☐

Example 2.4.3. The function $z \mapsto z: \mathbf{C} \to \mathbf{C}$ is complex differentiable by inspection of the limit (2.3.3.a). By Proposition 2.4.1, it follows that each rational function $f: U \to \mathbf{C}$ defined on an open subset $U \subseteq \mathbf{C}^n$ is complex differentiable.

Proposition 2.4.4. *Let $U \subseteq \mathbf{C}^n$ be an open subset. If $f: U \to \mathbf{C}$ is an analytic function, then f is complex differentiable.*

Proof. Let $w \in U$, and assume that f admits a power-series representation

$$f(z) = \sum_{n \geq 0} a_n (z-\zeta)^n$$

centered at ζ in some open neighborhood of w. The map

$$\varphi: z \mapsto z + \zeta: \mathbf{C} \to \mathbf{C}$$

is a biholomorphism, and the composite of two complex-differentiable functions is complex differentiable by Proposition 2.4.2. Replacing f by $f \circ \varphi$ and w by $w - \zeta$, we may therefore assume without loss of generality that f admits a power-series representation

$$f(z) = \sum_{n \geq 0} a_n z^n \tag{2.4.4.a}$$

centered at 0 in some neighborhood V of w. Choose $r \in \mathbf{R}_{>0}$ such that $\mathbf{D}(w, r) \subseteq V$. It suffices to prove the following assertions:

(1) the series

$$g(z) := \sum_{n \geq 0} n a_n z^{n-1}$$

converges on $\mathbf{D}(w, r)$; and

(2) for each $z \in \mathbf{D}(\zeta, r)$, we have

$$0 = \lim_{h \to 0} \left(\frac{f(z+h) - f(z)}{h} - g(z) \right).$$

Consider Claim 2.4.4. Let R denote the radius of convergence of the power series in (2.4.4.a). In particular, we have $R \geq r$.

□

2.5 Implicit function theorem

This section follows the treatment of [ZS75, pp. 139-145].

Formal implicit function theorem

Remark 2.5.1. The *Cauchy product* of two power series in the variable z is given by

$$\left(\sum_{r \in \mathbf{Z}_{\geq 0}} a_r z^r \right) \cdot \left(\sum_{s \in \mathbf{Z}_{\geq 0}} b_s z^s \right) = \sum_{s \in \mathbf{Z}_{\geq 0}} \left(\sum_{r=0}^{s} a_r b_{s-r} \right) z^s. \tag{2.5.1.a}$$

Lemma 2.5.2. *Consider a complex formal power series of the form*

$$f(z_1, \ldots, z_n) = \sum_{(r_1, \ldots, r_n) \in \mathbf{Z}^n_{\geq 0}} c_{r_1, \ldots, r_n} z_1^{r_1} \cdots z_n^{r_n}. \tag{2.5.2.a}$$

If $f(0, \ldots, 0) \neq 0$, then there exists a unique formal power series $g(z_1, \ldots, z_n)$ such that $f \cdot g = 1$.

Proof. We proceed by induction on n. If $n = 0$, then the claim is vacuous. Let $n > 0$ and write f in the form

$$f(z_1, \ldots, z_n) = \sum_{r \in \mathbf{Z}_{\geq 0}} a_r(z_1, \ldots, z_{n-1}) z_n^r,$$

where $a_r(z_1, \ldots, z_{n-1})$ is a formal power series in $z_1, \ldots, z_{n-1}$. In particular, we have

$$a_0(0, \ldots, 0) = f(0, \ldots, 0) \neq 0.$$

The inductive hypothesis therefore implies that a_0 admits a multiplicative inverse as a formal power series in $z_1, \ldots, z_{n-1}$. Replacing f by $c_{0,\ldots,0}^{-1} f$, we may assume without loss of generality that $f(0, \ldots, 0) = 1$. By (2.5.1.a), we seek a formal power series

$$g(z_1, \ldots, z_n) = \sum_{s \in \mathbf{Z}_{\geq 0}} b_s(z_1, \ldots, z_{n-1}) z_n^s$$

satisfying the relations

$$\delta_{0s} = \sum_{r=0}^{s} a_r(z_1, \ldots, z_{n-1}) b_{s-r}(z_1, \ldots, z_{n-1}),$$

where δ_{0s} is the Kronecker symbol. We solve for the coefficients b_s recursively as follows. For $s = 0$, we have the unique solution $b_0 = a_0^{-1}$. For $s > 0$, we have

$$0 = \sum_{r=0}^{s} a_r b_{s-r} \quad \Rightarrow \quad a_0 b_s = -\sum_{r=1}^{s} a_r b_{s-r} \quad \Rightarrow \quad b_s = -a_0^{-1} \sum_{r=1}^{s} a_r b_{s-r},$$

and this solution is unique. □

Theorem 2.5.3 (Formal implicit function theorem)**.** *Consider a complex formal power series* (2.5.2.a) *such that* $c_{0,\dots,0} = 0$ *and* $c_{0,\dots,0,1} \neq 0$*. There exist a unique formal power series* $u(z_1, \dots, z_n)$ *and a unique formal power series* $r(z_1, \dots, z_{n-1})$ *such that*

$$uf = z_n - r \tag{2.5.3.a}$$

and $u(0, \dots, 0) \neq 0$.

Proof. Observe that, if there exist formal power series $u \in \mathbf{C}[\![z, w]\!]$ and $r \in \mathbf{C}[\![z]\!]$ such that $uf = w - r$, then the constant term of u is nonzero, and u is therefore invertible. Indeed, if not, then uf contains no nonzero linear terms, as the constant term of f is zero. This contradicts the fact that $z_n - r$ contains the nonzero linear term z_n, as r is a power series in $z_1, \dots, z_{n-1}$.

The set of complex formal power series in $z_1, \dots, z_n$ forms a complex vector space with respect to the evident addition and scalar-multiplication operations. We will switch to a mixed multi-index notation with $z = (z_1, \dots, z_{n-1})$ and $w = z_n$. We have a $\mathbf{C}$-linear morphism

$$R\colon \sum_r c_r(z) w^r \mapsto c_0(z)\colon \mathbf{C}[\![z, w]\!] \to \mathbf{C}[\![z]\!],$$

where we identify elements of $\mathbf{C}[\![z, w]\!]$ with formal power series in the single variable w with coefficients $c_r(z) \in \mathbf{C}[\![z]\!]$.

We also have a $\mathbf{C}$-linear morphism

$$H\colon p \mapsto (p - R(p))w^{-1}\colon \mathbf{C}[\![z, w]\!] \to \mathbf{C}[\![z, w]\!].$$

In terms of coefficients, H is given by

$$H\Big(\sum_{r \geq 0} c_r(z) w^r\Big) = \sum_{r > 0} c_r(z) w^{r-1}.$$

In particular, we have the relation

$$p = wH(p) + R(p) \tag{2.5.3.b}$$

for each $p \in \mathbf{C}[\![z, w]\!]$.

It suffices to find a formal power series u in z such that

$$0 = 1 - uH(f) - H(uR(f)). \tag{2.5.3.c}$$

Indeed, this follows from the equalities

$$\begin{aligned}
H(w - uf) &= H(w) - H(uf) && \text{additivity of } H\\
&= 1 - H(uf) && \text{definition of } H\\
&= 1 - H(uwH(f) + uR(f)) && \text{(2.5.3.b)}\\
&= 1 - H(uwH(f)) - H(uR(f)) && \text{additivity of } H\\
&= 1 - (uwH(f) - R(uwH(f)))w^{-1} - H(uR(f)) && \text{definition of } H\\
&= 1 - uH(f) - H(uR(f)) && \text{definition of } R
\end{aligned}$$

and the observation that, by (2.5.3.b), $0 = H(w - uf)$ implies that $R(w - uf) = w - uf$ is a formal power series in w, and we have $w = uf + (w - uf)$, so u and $r \coloneqq w - uf$ satisfy the required relation $uf = w - r$.

For the sake of brevity, we set

$$\mu \coloneqq -R(f)H(f)^{-1}.$$

By Lemma 2.5.2, the hypothesis that $c_{0,\dots,0,1} \neq 0$ implies that $H(f)$ admits a multiplicative inverse in $\mathbf{C}[\![z,w]\!]$. It therefore suffices to find a formal power series v in z such that

$$0 = 1 - v + H(\mu v).$$

substituting $u = vH(f)^{-1}$ into (2.5.3.c). Equivalently, we seek a formal power series v in z such that

$$v = 1 + M(v), \tag{2.5.3.d}$$

where M is the $\mathbf{C}$-linear map

$$M\colon p \mapsto H(\mu p)\colon \mathbf{C}[\![z,w]\!] \to \mathbf{C}[\![z,w]\!].$$

By linearity of M, our solution v must satisfy

$$v = 1 + M(v) = 1 + M(1 + M(v)) = 1 + M(1) + M^2(v).$$

Applying (2.5.3.d) recursively, our solution must therefore satisfy

$$v = 1 + M(v) + \cdots + M^{k+1}(v) \tag{2.5.3.e}$$

for each $k \in \mathbf{Z}_{\geq 0}$.

The hypothesis that $c_{0,\dots,0} = 0$ implies that the constant term of $R(f)$ is zero. It follows that, if $p \in \mathbf{C}[\![z,w]\!]$ is a formal power series whose nonzero terms are divisible by at least one of the monomials

$$z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} \tag{2.5.3.f}$$

of total degree $\geq r$, i.e., with $(\alpha_1, \dots, \alpha_{n-1}) \in \mathbf{Z}_{\geq 0}^{n-1}$ such that $\sum_k \alpha_k \geq r$, then each nonzero term of $M(p)$ is divisible by at least one of the monomials (2.5.3.f) of total degree $\geq r+1$. In particular, it follows by induction that a monomial of the form (2.5.3.f) of total degree $\geq k$ divides each nonzero constant term of $M^k(1)$. It follows that formal power series of formal power series

$$v = \sum_{k \in \mathbf{Z}_{\geq 0}} M^k(v) \tag{2.5.3.g}$$

converges to a formal power series in $\mathbf{C}[\![z,w]\!]$.

By (2.5.3.e), each solution v must satisfy the relation (2.5.3.g), so the solution v is unique if it exists.

Let us show that (2.5.3.g) is indeed a solution. Let $k \in \mathbf{Z}_{\geq 0}$. We have

$$v = 1 + M(1) + \cdots + M^k(1) + W_k,$$

where we regard W_k is regarded as a power series in w with coefficients in $\mathbf{C}[\![z]\!]$. By our previous observation, each nonzero term of W_k is divisible by a monomial (2.5.3.f) of total degree $\geq k+1$. For each $k \in \mathbf{Z}_{\geq 0}$, we have

$$\begin{aligned} v - 1 - M(v) &= \left(W_k + \sum_{\ell=0}^{k} M^\ell(1)\right) - 1 - \left(M(W_k) + \sum_{\ell=1}^{k+1} M^\ell(1)\right) \\ &= W_k - M^{k+1}(1) - M(W_k). \end{aligned}$$

It follows that each nonzero term of $v - 1 - M(v)$ is divisible by a monomial (2.5.3.f) of total degree $\geq k+1$. As this is true for each $k \in \mathbf{Z}_{\geq 0}$, it follows that $v - 1 - M(v) = 0$. $\square$

Remark 2.5.4. Explain why this is an implicit function theorem.

Analytic implicit function theorem

Theorem 2.5.5 (Single-variable analytic inverse function theorem)**.** *Let $U \subseteq \mathbf{C}$ be an open subset, let $f\colon U \to \mathbf{C}$ be an analytic function on U, and let $a \in U$. If f is analytic at a with power series expansion*

$$f(z) = \sum_{n \in \mathbf{Z}_{\geq 0}} c_n z^n$$

and the coefficient c_1 is nonzero, then f admits a local analytic inverse function at a.

Proof. We will give a proof of the several-variable analogue of this statement later. In the single-variable case, it is not difficult to write down a formal inverse power series and check that it converges near a. See [Lan99, Chapter II, Theorem 6.1] for details. □

2.6 Key theorems of complex analysis

Much of this section follows the exposition of [Lan99, Chapter II] and [Sch05, §1.2.2].

Lemma 2.6.1. *Let $U \subseteq \mathbf{C}^n$ be an open subset, let $z \in U$, let $w \in \mathbf{C}^n$, and let*

$$V \coloneqq \{c \in \mathbf{C} \mid z + cw \in U\} \subseteq \mathbf{C}.$$

(1) *The subset V is an open neighborhood of* 0*.*

(2) *For each analytic function $f\colon U \to \mathbf{C}$ an analytic function, the function $g\colon V \to \mathbf{C}$ given by $g(c) = f(z + cw)$ is analytic.*

Proof. Consider Claim (1). The element $z = z + 0 \cdot w$ belongs to U, so $0 \in V$. Translating everything by $-z$, we may assume without loss of generality that $z = 0$. In this case, V is the preimage of the open set U under the continuous map $t \mapsto tw\colon \mathbf{C} \to \mathbf{C}^n$.

Consider Claim (2). The map $t \mapsto z + tw$ is analytic, and g is the composite of this map with the analytic function f, and is therefore itself analytic. □

Theorem 2.6.2 (Identity theorem)**.** *Let $\varnothing \neq V \subseteq U \subseteq \mathbf{C}^n$ be open subsets and assume that U is connected.*

(1) *If $f\colon U \to \mathbf{C}$ is an analytic function such that $f|_V = 0$, then $f = 0$.*

(2) *If $f, g\colon U \to \mathbf{C}$ are analytic functions such that $f|_V = g|_V$, then $f = g$.*

Proof. The two assertions are equivalent: on the one hand, Claim (2) follows from Claim (1) as the difference $f - g$ is analytic if f and g are (Exercise 2.1.13.(1)), so Claim (1) implies that $f - g = 0$; on the other hand, Claim (1) is the special case of Claim (2) in which $g = 0$. We shall prove Claim (1).

The topological space $\mathbf{C}$ is Hausdorff by Example 1.3.15, so the singleton $\{0\} \subseteq \mathbf{C}$ is closed. By Exercise 2.1.12, f is continuous. The preimage $\bar{Z} \coloneqq f^{-1}(\{0\}) \subseteq U$ is therefore closed. Let Z denote the interior of $\bar{Z}$, which is open (Definition 1.1.12.(2)). We will show that the complement $\partial Z = \bar{Z} - Z$ is empty and, hence, that Z is also closed. As Z is therefore open, closed, and nonempty by virtue of the inclusion $V \subseteq Z$, connectedness of U implies that $Z = U$ (Proposition 1.3.4) and, hence, that $f = 0$.

Suppose for contradiction that $a \in \partial Z$. By Exercise 1.1.4, we may choose an open polydisk D centered at a and contained in U. It suffices to show that $D \subseteq Z$ and, hence, that $a \in \partial Z \cap D \subseteq \partial Z \cap Z = \varnothing$.

As D is open, in order to show that $D \subseteq Z$, it suffices to show that $f(z) = 0$ for each $z \in D$. The open subset D of U meets the boundary ∂Z, so it also meets the interior Z by Proposition 1.1.14. Let $w \in D \cap Z$ and let $z \in D$. The subset

$$W \coloneqq \{c \in \mathbf{C} \mid w + c(z - w) \subseteq U\} \subseteq \mathbf{C}$$

is open by Lemma 2.6.1.(1), and the function

$$g\colon t \mapsto f(w + t(z - w))\colon W \to \mathbf{C}$$

is analytic by Lemma 2.6.1.(2). By definition of Z, f vanishes identically on an open neighborhood of w, so g vanishes identically on the preimage of this open neighborhood in W, which is an open neighborhood of 0. By Exercise 1.1.6, D is convex, so $[0, 1] \subseteq W$, and, by Proposition 1.3.2, $[0, 1]$ is connected. By the identity theorem for single-variable analytic functions, g is thus identically zero on $[0, 1]$. Thus, $g(1) = f(w + 1 \cdot (z - w)) = f(z) = 0$, as required. □

Theorem 2.6.3 (Liouville)**.** *If $f\colon \mathbf{C}^n \to \mathbf{C}$ is a bounded, analytic function, then f is constant.*

Proof. Let $z, w \in \mathbf{C}^n$. Consider the function $h\colon \mathbf{C} \to \mathbf{C}^n$ given by $c \mapsto z + cw$. This function is affine and *a fortiori* analytic, so the composite $g := f \circ h$ is analytic. The image of g is contained in the image of f, so g is bounded. By the single-variable Liouville Theorem, g is constant, so we have

$$f(z) = g(0) = g(1) = f(w),$$

which shows that f is also constant. □

Theorem 2.6.4 (Maximum modulus principle)**.** *Let $U \subseteq \mathbf{C}^n$ be a connected open subset, let $f\colon U \to \mathbf{C}$ be an analytic function, let $a \in U$. If $|f(a)| \geq |f(z)|$ for each z in an open neighborhood $a \in V \subseteq U$, then f is constant on U.*

Proof. By the Identity Theorem (Theorem 2.6.2.(2)), if f is constant on V, then f is also constant on U, so it suffices to show that f is constant on V. We may furthermore replace V by an open polydisk

$$D = \mathbf{D}(a_1, r_1) \times \cdots \times \mathbf{D}(a_n, r_n)$$

centered at a contained in V. The function

$$z_1 \mapsto f(z_1, a_2, \ldots, a_n)$$

is analytic as a composite of analytic functions, and its modulus attains its maximum at a_1. The function f_1 is therefore constant on $\mathbf{D}(a_1, r_1)$ with value $f(a)$ by the single-variable maximum modulus principle.

Suppose that the function

$$(z_1, \ldots, z_k) \mapsto f(z_1, \ldots, z_k, a_{k+1}, \ldots, a_n)$$

is constant on $\mathbf{D}(a_1, r_1) \times \cdots \times \mathbf{D}(a_k, r_k)$ for some $1 \leq k < n$. For each $(w_1, \ldots, w_k)$ in this polydisk, the function

$$z_{k+1} \mapsto f(w_1, \ldots, w_k, z_{k+1}, a_{k+2}, \ldots, a_n)$$

is analytic on $\mathbf{D}(a_{k+1}, r_{k+1})$ and its modulus attains its maximum at a_{k+1}, so the function is constant with value $f(a)$ on this disk. By induction, f is constant on the polydisk. □

Theorem 2.6.5 (Open Mapping Theorem)**.** *Let $U \subseteq \mathbf{C}^n$ be a connected open subset and let $f\colon U \to \mathbf{C}$ be a nonconstant analytic function. For each open subset $V \subseteq U$, the image $f(V) \subseteq \mathbf{C}$ is open.*

Proof. Let $z = (z_1, \ldots, z_n) \in V$ and let $w = f(z)$. We will construct an open neighborhood of w contained in $f(V)$ and the claim will follow. By Exercise 1.1.4, as V is open, there exists a polydisk D centered at z and contained in V. As V is covered by such polydisks, and as open sets are stable under arbitrary unions, we may assume without loss of generality that $V = D$.

As f is nonconstant, we may choose $z' \in D$ such that $f(z) \neq f(z')$. As in Lemma 2.6.1, consider the open neighborhood

$$W := \{c \in \mathbf{C} \mid z + c \cdot (z' - z) \in D\}$$

of the origin in $\mathbf{C}$, and the analytic map $h\colon c \mapsto z + cz' \colon W \to D$. The composite $g := f \circ h$ is analytic and nonconstant by construction. By the single-variable Open Mapping Theorem, g is an open map. In particular, the image of g is an open neighborhood of w contained in $f(D)$, as required. $\square$

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