Topology of algebraic varieties

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Wintersemester 2019/2020

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11.1 Nori's approach

We have often used the fact that every cell complex has a *skeletal filtration*, i.e.,

$$Y_0 \subset Y_1 \subset \cdots \subset Y_n = Y$$

such that

$$H^i(Y_j, Y_{j-1}) = \begin{cases} \mathbb{Z}^N & i = j \\ 0 & i \neq j \end{cases}$$

We have seen that affine algebraic varieties are homotopy equivalent to a cell complex, so they admit a skeletal filtration. Surprisingly, this filtration can be realised by *affine subvarieties* (defined over the ground field if we are working over fields different from $\mathbb{C}$).

Theorem 11.1.1 (Basic Lemma (Nori 2002, Beilinson 1987, Vilonen ?))**.** *Let $X \subset \mathbb{A}^n$ be an affine variety, $Z \subsetneq X$ a closed subvariety. Then there is a closed subvariety $Z \subset Y \subsetneq X$ such that*

$$H^i(X, Y) = \begin{cases} \mathbb{Z}^N & i = \dim(X) \\ 0 & \textit{else.} \end{cases}$$

Example 11.1.2. Assume $X = \bar{X} - H$ for a non-singular projective $\bar{X}$ and a transversal hyperplane section H. Choose a second hyperplane section H' transversal to X and H. Put $Y = X \cap H'$. We claim that it satisfies the property of the theorem. By Bertini's theorem H, H' and Y are non-singular. As X and Y are both non-singular affine, our version of Artin vanishing shows that their cohomology vanishes above $\dim X$.

Proof. Hence $H^i(X, Y) = 0$ for $i > \dim(X)$.

We have Gysin sequences for $(\bar{X}, H)$ and $(H', H \cap H')$:

$$\begin{array}{ccccccccc} \dots \longrightarrow & H^{i-2}(H) & \longrightarrow & H^i(\bar{X}) & \longrightarrow & H^i(X) & \longrightarrow & H^{i-1}(H) & \longrightarrow \dots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \dots \longrightarrow & H^{i-2}(H' \cap H) & \longrightarrow & H^i(H') & \longrightarrow & H^i(Y) & \longrightarrow & H^{i-1}(H' \cap H) & \longrightarrow \dots \end{array}$$

Let $d=\dim(X)$. By the Lefschetz hyperplane theorem, we have isomorphisms for $i<d-1$ and $i-1<\dim(H)-1=d-2$. Hence we also have isomorphisms $H^i(X)\to H^i(Y)$ for $i<d-1$. Moreover, we have injections for $i=d-1$. A diagram chase shows that we also have an injection for $H^{d-1}(X)\to H^{d-1}(H)$. Hence $H^i(X,Y)=0$ for $i<d-1$. That it is even free follows from the universal coefficient theorem because the vanishing is true for any coefficients. □

We deduce the "skeletal filtration"

Corollary 11.1.3. *Let X be an affine algebraic variety of dimension d. Then there is an sequence*

$$X_0\subset X_1\subset\cdots\subset X_d=X$$

of closed subvarieties such that

$$H^i(X_j,X_{j-1})=\begin{cases}\mathbb{Z}^N & i=j\\ 0 & else\end{cases}$$

Moreover, the X_j can be chosen of dimension j and such that X_j-X_{j-1} is non-singular.

Proof. Let Z be a codimension 1 subvariety of X containing the singular locus of X. Let Y_{d-1} be the subvariety produced by the Basic Lemma. Proceed by induction. □

Proof of the Basic Lemma. We follow [HMS17] p. 48. The argument is due to Nori. We first work with coefficients in a field. Without loss of generality Z contains all singularities of X. By Artin vanishing (in the general form) we have $H^i(X,Y)=0$ for $i>d$ and any subvariety of X containing Z. It remains to choose Y such that we also have vanishing for $i<d$.

Let $\bar{X}$ be projective closure of X. By resolution of singularities there is a blow-up $\pi:\tilde{X}\to\bar{X}$ (proper surjective, isomorphism over $X-Z$) such that $\tilde{X}$ is non-singular and both the preimages $\tilde{Z}$ of $\bar{Z}$ and D_∞ of $\bar{X}-X$ and even $\tilde{Z}\cup D_\infty$ are divisor with normal crossings (locally in the analytic topology given by equations $z_1\dots z_m=0$). We choose a generic hyperplane section $\tilde{H}$ such that $\tilde{Z}\cup D_\infty\cup\tilde{H}$ is still a divisor with normal crossings. (This is again a version of Bertini's theorem.) Put $D_0=\tilde{Z}\cup\tilde{H}$ and $Y=\pi(D_0-D_\infty\cap D_0)$. This is a closed subvariety of X and contains Z. By excision,

$$H^i(\tilde{X}-D_\infty,D_0\cap D_\infty)\rightsquigarrow H^i(X,Y),$$

hence it suffices to show vanishing on the level of $\tilde{X}$. By a version of Poincaré duality (see below), we have

$$H^i(\tilde{X}-D_\infty,D_0-D_0\cap D_\infty)\rightsquigarrow H^{2d-i}(\tilde{X}-D_0,D_\infty-D_0\cap D_\infty)^*$$

As complement of a hyperplane, the variety $\tilde{X}-D_0$ is affine. By Artin vanishing (the non-singular version this time), the right hand side vanishes for $2d-i>d$, hence for $d>i$. The only remaining cohomology is in degree d.

Playing around with the universal coefficient theorem implies that even integral cohomology is concentrated in degree d and even free. □

Theorem 11.1.4 (Poincaré duality)**.** *Let X be a non-singular projective variety, D_0 and D_∞ divisors with normal crossings such that $D_0\cup D_\infty$ is again a divisor with normal crossings. Then the duality map induces isomorphisms*

$$H^i(X-D_0,D_\infty-D_\infty\cap D_0)\rightsquigarrow H^{2d-i}(X-D_\infty,D_0-D_\infty\cap D_0)^*$$

in singular cohomology with coefficients in a field.

Remark 11.1.5. The intersection condition on D_0 and D_∞ is necessary. There is a counterexample in [HMS17, Remark 2.4.6].

Proof. The complete argument is above our technical level. The result follows from patching together local isomorphisms similar to our proof of Poincaré duality. Locally, we have coordinates $z_1, \dots, z_d$ and D_0 is given by $z_1 \dots z_{n_0} = 0$ and D_∞ by $z_{n_0+1} \dots z_{n_\infty} = 0$. Let U_0 be a polydisk with coordinates $z_1, \dots, z_{n_0}$, U_∞ a polydisk with coordinates $z_{n_0+1}, \dots, z_{n_\infty}$. Let Z_0 and Z_∞ be the intersection with the divisors. Locally, we are in a product situation $U_0 \times U_\infty$ with divisors $Z_0 \times U_\infty$ and $U_0 \times Z_\infty$. Roughly we get our isomorphism from local Poincaré duality isomorphisms

$$H_i(U_0 - Z_0) \cong H^i(U_0, Z_0) \text{ and } H_j(U_\infty - Z_\infty) \cong H^j(U_\infty, Z_\infty).$$

□

11.2 The sheaf theoretic approach

One way to view homology is to see it as a means of keeping track of how local invariants induce global ones. For example, a manifold is locally trivial (and locally its homology vanishes), but globally non-trivial (its homology does not vanish). Sheaves are a systematic way of organising the transition.

Definition 11.2.1. Let X be a topological space. A *presheaf* of abelian groups on X assigns to every open subset $U \subset X$ an abelian group $\mathcal{F}(U)$

$$U \mapsto \mathcal{F}(U)$$

and to every inclusion $V \subset U$ a group homomorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ (the restriction homomorphism) such that $\rho_{UU} = \mathrm{id}$, $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ for all $W \subset V \subset U$. A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ consists of homomorphisms $\mathcal{F}(U) \to \mathcal{G}(U)$ compatible with the restriction maps.

A presheaf is a *sheaf* if for every open cover $U = \bigcup_{i \in I} U_i$ the sequence

$$0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j)$$

(where the map on the right is the difference of the restriction maps) is exact. A morphism of sheaves is a morphism in the category of presheaves.

Example 11.2.2. (1) Let A be an abelian group. The assignment $U \mapsto A$ is a presheaf (the *constant presheaf*), but in general not a sheaf.

(2) Let X be a complex manifold, $\mathcal{O}(U)$ the ring of holomorphic functions. This is a presheaf and even a sheaf.

Lemma 11.2.3. *Let X be topological space, $\mathcal{F}$ a presheaf on X. There is a unique sheaf $\mathcal{F}^+$ (the* sheafification*) together with a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ such that every morphism $\mathcal{F} \to \mathcal{G}$ into a sheaf factors uniquely via θ.*

Proof. Omitted. □

Example 11.2.4. The *constant sheaf* is the sheafification of the constant presheaf. If X is a manifold, $U = \bigcup_{i \in I} U_I$ a disjoint union of connected open subsets, then

$$U \mapsto A^I.$$

In particular, $U \mapsto A$ for connected open sets U.

Definition 11.2.5. Let $f : X \to Y$ be a continuous map of topological spaces, $\mathcal{F}$ a sheaf on X. We define a sheaf $f_*\mathcal{F}$ on Y by

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)).$$

It is called *push-forward* of $\mathcal{F}$.

Let $\mathcal{G}$ be a sheaf on Y. We define a sheaf $f^*\mathcal{F}$ on X as the sheafification of

$$f^*\mathcal{G}(U) = \lim_{V \supset f(U)} \mathcal{G}(V).$$

If $j : W \to X$ is an open embedding, we also define $j_!\mathcal{F}$ as the sheafification of

$$j_!\mathcal{F}(U) = \begin{cases} \mathcal{F}(U) & U \subset W \\ 0 & \text{else.} \end{cases}$$

If $Y \subset X$ is a subspace, we also write $\mathcal{F}|_Y$ for the pull-back.

Remark 11.2.6. For good topological spaces like manifolds of the analytic spaces attached to algebraic varieties, there is a way of extending the definition of singular cohomology to cohomology with coefficients in sheaves $H^i(X, \mathcal{F})$. If $j : U \to X$ is an open embedding, then

$$H^i(X, j_!\mathbb{Z}) \cong H^i(X, X - U; \mathbb{Z}).$$

Now we are ready to define our main player:

Definition 11.2.7. Let X be an algebraic variety over $\mathbb{C}$. A sheaf on X^{an} is called *weakly constructible* if there is a stratification of X into locally closed subvarieties $(S_i)_{i \in I}$ such that the $\mathcal{F}|_{S_i^{\text{an}}}$ are locally constant.

The strata S_i can be chosen non-singular.

Theorem 11.2.8 (Basic Lemma II). *Let X be an affine variety of dimension n and $\mathcal{F}$ a weakly constructible sheaf on X. Then there exists a Zariski open subset $j : U \to X$ such that*

(1) $\dim(X - U) < n$,
(2) $H^q(X, \mathcal{F}') = 0$ *for* $q \neq n$ *where* $F' = j_! j^* \mathcal{F} \subset \mathcal{F}$,

Lemma 11.2.9. *Basic Lemma II implies the Basic Lemma.*

Proof. Use $\mathcal{F} = j_{Z!}\mathbb{Z}$ for $j_Z : X - Z \to X$. □

References

[HMS17] Annette Huber and Stefan Müller-Stach. *Periods and Nori motives*, volume 65 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer, Cham, 2017. With contributions by Benjamin Friedrich and Jonas von Wangenheim.