Topology of algebraic varieties

Brad Drew and Annette Huber

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10 Affine varieties

10.1 A Morse function

Our aim is to prove the following result:

Theorem 10.1.1. *Let X non-singular, affine algebraic variety of dimension d, i.e., a non-singular algebraic variety given by a Zariski-closed subset $X \subset \mathbf{C}^n$ for some $n \in \mathbf{Z}_{\geq 0}$. Then X is deformation equivalent to a finite cell complex of dimension d.*

Note that X has real dimension $2d$.

Corollary 10.1.2 (Artin vanishing)**.** *The homology and cohomology of X are finitely generated and concentrated in degree at most d.*

Proof. This is true for all finite cell complexes of dimension at most d: use induction for the skeletal filtration and the corresponding long exact sequences, see Remark VIII.2.8. In the induction step, we consider

$$\dots H_i(X_{n-1}, \mathbb{Z}) \to H_i(X_n, \mathbb{Z}) \to H_i(X_n, X_{n-1}; \mathbb{Z}) \to \dots$$

Hence $H_i(X_n, \mathbb{Z})$ is an extension of a submodule $H_i(X_{n-1}, \mathbb{Z})$ (hence finitely generated, 0 for $i > n-1$) and a submodule of

$$H_i(X_n, X_{n-1}; \mathbb{Z}) = \begin{cases} \mathbb{Z}^{\# n\text{-cells}} & i = n \\ 0 & i \neq n \end{cases}.$$

□

Remark 10.1.3. The Theorem was first proved by Andreotti and Frankel for *Stein manifolds*. These are complex manifolds with many holomorphic sections, e.g., defined as the vanishing locus of holomorphic functions on $\mathbb{C}^n$. In this case, X is a deformation retract of a CW-complex of dimension $\leq d$.

The theorem also holds for singular Stein spaces, in particular also for singular affine varieties. We follow the presentation of Voisin, see [Voi07, Chapter 1.2].

The proof uses again Morse theory, but not for the real part of a holomorphic function.

Let $h : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be the standard hermitian form. Note that $s = \Re(h)$ is the standard scalar product on $\mathbb{R}^{2n}$: For $n = 1$:

$$\Re(h(x, y)) = \Re(x\bar{y}) = \Re(x)\Re(y) - \Im(x)(-\Im(y)).$$

Definition 10.1.4. Let $X \subset \mathbb{R}^N$ be a smooth submanifold. For fixed $P \in \mathbb{R}^n$ we define

$$f = f_P : X \to \mathbb{R}, \qquad f(x) = s((x - P), (x - P)),$$

the square of the distance from x to P.

This function is obviously smooth on $\mathbb{R}^N$, hence on X.

Lemma 10.1.5. *The function is exhaustive.*

Proof. The function takes values in $[0, \infty)$. For every $M \in [0, \infty]$, the preimage $f^{-1}([0, M])$ is bounded and closed, hence compact. □

Is it a Morse function, i.e., does it have only non-degenerate critical points?

Example 10.1.6. Let $S^1 \subset \mathbb{R}^2$ be the unit sphere and use $P = (-1, 0)$. For $Q = (x, y) \in S^1$, we have

$$f(Q) = (x + 1)^2 + y^2.$$

In order to determine the critical points, we have to use local coordinates on S^1. We parametrize $(x, \pm\sqrt{1 - x^2})$ for the upper or lower semi-circle. Hence

$$f(Q) = (x + 1)^2 + (1 - x^2) = 2x + 2.$$

Its derivative does not vanish. Alternatively, we parametrize $(\pm\sqrt{1 - y^2}, y)$ for the right or left semi-circle. Hence

$$f(Q) = (\pm\sqrt{1 - y^2} + 1)^2 + y^2 = 1 - y^2 \pm 2\sqrt{1 - y^2} + 1 + y^2 = 2 \pm 2\sqrt{1 - y^2}.$$

Its derivative is

$$\pm\frac{-2y}{\sqrt{1 - y^2}},$$

hence we get critical points for $y = 0$. These are $(-1, 0), (1, 0)$. They did not show up in the other charts. Hence the critical points are isolated. We compute the Hessian, i.e., the second derivative:

$$\pm\frac{-2\sqrt{1 - y^2} + 2y(-2y)\frac{1}{2}\sqrt{1 - y^2}^{-1}}{1 - y^2}$$

It takes values ± 2, hence the critical point is non-degenerate. The function is a Morse function.

However, this is not always the case:

Example 10.1.7. Let $X = S^1 \subset \mathbb{R}^2$, $P = (0, 0)$. By definition

$$f(x, y) = x^2 + y^2 = 1,$$

hence all points are critical. They are not isolated.

Note that the level sets X_M for our function are the intersections of X with a sphere of radius $\sqrt{M}$ around P. Hence the points are critical, if the level set touches this ball, i.e., if the tangent space of X_M is contained in the tangent space of the sphere. This is equivalent to the tangent space being perpendicular to the vector $\overrightarrow{PQ}$.

Lemma 10.1.8. *Let $X \subset \mathbb{R}^N$ be a smooth submanifold, fix $P \in \mathbb{R}^N$. Let $Q \in X$. Then*

$$df_{P,Q} : T_{X,Q} \to \mathbb{R}$$

is given by

$$df_{P,Q}(u) = 2s(\overrightarrow{PQ}, u).$$

In particular, $Q \in X$ is a critical point of f_P if and only if $\overrightarrow{PQ}$ is orthogonal to $T_{X,Q}$ in $\mathbb{R}^N$.

Proof. We check the formula in local coordinates. Let $x_1, \dots, x_N$ be the standard coordinates on $\mathbb{R}^N$. Without loss of generality, $x_1, \dots, x_d$ are are coordinates for X near $Q = (q_1, \dots, q_N)$. There are smooth functions $g_{d+1}(x_1, \dots, x_d), \dots, g_N(x_1, \dots, x_d)$ such that the points in this coordinate chart are given as

$$(x_1, \dots, x_d, g_{d+1}(x_1, \dots, x_d), \dots, g_N(x_1, \dots, x_d))$$

The standard tangent vector of X in Q in direction x_1 is identified with

$$u_1 = (1, 0, \dots, 0, \frac{\partial g_{d+1}}{\partial x_1}(q_1, \dots, q_d), \dots)$$

We write $P = (p_1, \dots, p_N)$, $Q = (q_1, \dots, q_N)$. Hence

$$f_P(x) = \sum_{i=1}^{d}(x_i - p_i)^2 + \sum_{i=d+1}^{N}(g_i(x_1, \dots, x_d) - p_i)^2.$$

Hence

$$\frac{\partial f_P}{\partial x_1} = 2(x_1 - p_1) + \sum_{i=d+1}^{N} 2(g_i(x) - p_i)\frac{\partial g_i}{\partial x_1}$$

This evaluates to

$$\frac{\partial f_P}{\partial x_1}(Q) = 2(q_1 - p_1) + \sum_{i=d+1}^{N} 2(q_i - p_i)\frac{\partial g_i}{\partial x_i}(Q) = 2s(Q - P, u_1).$$

The same computation works for the other basis vectors of $T_{X,Q}$. □

The idea is to vary P in order to find f_P where the critical points are non-degenerate.

Definition 10.1.9. Let $X \subset \mathbb{R}^N$ be a smooth submanifold. We put

$$Z = \{(Q, P) \in X \times \mathbb{R}^N | \overrightarrow{PQ} \perp T_{X,Q}\}.$$

Lemma 10.1.10. *Z is a smooth manifold of dimension N.*

Proof. We use the same coordinates near a point Q as in the last proof. The vector fields $u_1, \dots, u_d$ are a basis of the tangent bundle of X near Q. Hence Z is cut out by the equations

$$s(\overrightarrow{Px}, u_1), \dots, s(\overrightarrow{Px}, u_d)$$

in $\mathbb{R}^d \times \mathbb{R}^N$. Explicitly, the system of equations is

$$(x_1 - p_1) + \sum_{i=d+1}^{N}(g_i(x) - p_i)\frac{\partial g_i}{\partial x_1}(x) = 0$$

$$(x_2 - p_2) + \sum_{i=d+1}^{N}(g_i(x) - p_i)\frac{\partial g_i}{\partial x_2}(x) = 0$$

$$\dots$$

$$(x_d - p_d) + \sum_{i=d+1}^{N} (g_i(x) - p_i) \frac{\partial g_i}{\partial x_d}(x) = 0$$

The Jacobian of this system has rank d because the partial derivatives with respect to the variables $p_1, \ldots, p_d$ are linearly independent. Hence they cut out a submanifold of dimension $N + d - d$. $\square$

Lemma 10.1.11. *Let $(Q, P) \in Z$, $u \in T_{X,Q}$. Then $(u, 0) \in T_{X,Q} \times T_{\mathbb{R}^N,P}$ lies in $T_{Z,(Q,P)}$ if and only if $u \in T_{X,Q}$ is in the kernel of the quadratic form* $\mathrm{Hess}(f_P)$.

Proof. We use an alternative description of Z. Consider the cotangent bundle T_X^*. Let

$$\sigma : X \times \mathbb{R}^N \to T_X^* \times \mathbb{R}^N$$

be the section given by $(Q, P) \mapsto ((df_P)_Q, P)$. Its zero locus is Z by the formula for df_P. In local coordinates on X, the section σ can be written as

$$\sum_{i=1}^{d} \frac{\partial f_P}{\partial x_i} dx_i.$$

The tangent space to Z at (Q, P) is thus described as

$$T_{Z,(Q,P)} = \left\{ (u, w) \in T_{X,Q} \times T_{\mathbb{R}^N,P} | \partial_u \frac{\partial f_P}{\partial x_i}(Q, P) + \partial_w \frac{\partial f_P}{\partial x_i}(Q, P) = 0, i = 1, \ldots, d \right\}.$$

We write $u = \sum_{j=1}^{d} u_j \partial_{x_j}$. Thus the vector $(u, 0)$ is in $T_{Z,(Q,P)}$ if and only if for $i = 1, \ldots, d$

$$\sum_{j=1}^{d} u_j \frac{\partial}{\partial x_j} \frac{\partial f_P}{\partial x_i}(Q, P) = 0.$$

By definition this means that u is in the kernel of $\mathrm{Hess} f_P$. $\square$

Corollary 10.1.12. *For fixed P, the point $(Q, P) \in Z$ is a degenerate critical point of f_P if and only if $\pi : Z \to \mathbb{R}^N$ is not an immersion.*

Proof. The elements in the kernel of $d\pi : T_Z \to \mathbb{R}^N$ are the ones of the form $(u, 0) \in T_X \times \mathbb{R}^N$. $\square$

Proposition 10.1.13. *For general $P \in \mathbb{R}^N$, the map f_P is a Morse function.*

Proof. Let Σ be the set of points of Z where π is not an immersion. Note that $\dim Z = \dim \mathbb{R}^N$, hence π is an immersion at (Q, P) if and only it is a submersion. By Sard's Theorem (see Theorem 6.1.3 or Theorem 10.3.1), the image $\pi(\Sigma)$ has Lebesgue measure 0. For every point $P \notin \pi(\Sigma)$, the map f_P is a Morse function. $\square$

10.2 The complex case

From now on we fix P such that f_P is a Morse function. We now specialise to the case $X \subset \mathbb{C}^n$. We have

$$f(Q) = h(\overrightarrow{PQ}, \overrightarrow{PQ})$$

for the standard hermitian form h on $\mathbb{C}^n$.

Proposition 10.2.1. *If $X \subset \mathbb{C}^n$ is algebraic, the f_P has only finitely many critical points and X has a deformation retract to a finite cell complex.*

Proof. With the notation used before, the critical points are the fibre of $\pi : Z \to \mathbb{R}^N$ in P. By choice of P, the map π induces an isomorphism on tangent spaces in all points of $\pi^{-1}(P)$. This extends to a neighbourhood of $f^{-1}(P)$, hence $f^{-1}(P)$ is discrete. The set $Z \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ is defined by polynomial equations because $X \subset \mathbb{C}^n$ is algebraic and $s : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ as well. This makes $f^{-1}(P) \to P$ a discrete algebraic set, hence finite.

We now apply Morse theory, see Theorem VI.3.3. Hence every $X_{\leq M}$ is a deformation retract of a finite cell complex. As f_P has only finitely many critical points, there is M such that $X_{\geq M} \to [M, \infty)$ does not have any critical points, making it a trivial fibration. Hence $X_{\leq M}$ is a deformation retract of X. □

In order to prove the Theorem of Andreotti-Frankel 10.1.1, it remains to compute the Morse index for our critical points. This is a local computation that needs some preparation.

Definition 10.2.2. Let $X \subset \mathbb{R}^N$ be a smooth submanifold, $Q \in X$. We define the *second fundamental form*

$$\Phi : T_{X,Q} \times T_{X,x} \to \mathbb{R}^N / T_{X,Q}$$

as follows: For $v \in T_{X,Q}$ choose a vector field V on a neighbourhood U of Q such that $V(Q) = v$. Via $T_{X,Q} \subset \mathbb{R}^N$, we can view V as a smooth map $V : U \to \mathbb{R}^N$. For $u \in T_{X,Q}$ we put

$$\Phi(u, v) = \partial_u V \mod T_{X,Q}.$$

Lemma 10.2.3. Φ *is well-defined, symmetric and bilinear. If* $X \subset \mathbb{C}^n$ *is a complex submanifold, then it is even* $\mathbb{C}$*-bilinear.*

Proof. We compute. Let $x_1, \ldots, x_N$ be the standard coordinates on $\mathbb{R}^N$. Without loss of generality $x_1, \ldots, x_d$ are coordinates for X near Q. This means that there is an open neighbourhood $U \subset X$ of Q such that $(x_1, \ldots, x_d) : U \to U' \subset \mathbb{R}^d$ is a diffeomorphism. Then $\partial_i = \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, d$ are a basis of TU. The map $U' \to \mathbb{R}^N$ has the shape

$$(x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_d, g_{d+1}(x_1, \ldots, x_d), \ldots, g_N(x_1, \ldots, x_d))$$

hence the images of ∂_i in $T_{\mathbb{R}^N}$ are the vector fields

$$V_1 = (1, 0, \ldots, 0, \frac{\partial g_d}{\partial x_1}, \ldots, \frac{\partial g_N}{\partial x_1})$$
$$V_2 = (0, 1, \ldots, 0, \frac{\partial g_d}{\partial x_2}, \ldots, \frac{\partial g_N}{\partial x_2})$$
$$\ldots$$

Hence

$$V = \sum_{i=1}^d a_i V_i$$

for smooth functions $a_i : U' \to \mathbb{R}$. Hence

$$\Phi(\partial_j, V) = \partial_j \sum_{i=1}^d a_i V_i = \sum_{i=1}^d \frac{\partial a_i}{\partial x_j}(Q) V_i(Q) + \sum_{i=1}^d a_i(Q) \frac{\partial V_i}{\partial x_j}(Q) \equiv \sum_{i=1}^d a_i(Q) \frac{\partial V_i}{\partial x_j}(Q)$$

This shows well-definedness. Bilinearity is obvious. For symmetry, we specialise to $V = V_i$. Then the above is equal to

$$\frac{\partial V_i}{\partial x_j}(Q) = (0, \ldots, 0, \frac{\partial^2 g_d}{\partial x_j \partial x_i}(Q), \ldots, \frac{\partial^2 g_N}{\partial x_j \partial x_i}(Q)),$$

hence symmetric.

If $X \subset \mathbb{C}^n$, then the same computation as above can be done with holomorphic coordinates and a $\mathbb{C}$-basis of the tangent space. We get same formula, which is still bilinear and symmetric. □

Remark 10.2.4. Φ has more canonical interpretation as the differential of the Gauß map $X \to \mathrm{Grass}(d, N)$. This does not seem to simplify the verification of the properties.

Proposition 10.2.5. *Let $X \subset \mathbb{R}^N$ be a smooth submanifold. Let $P \in \mathbb{C}^n$ such that f_P is a Morse function. Let $Q \in X$ be a critical point. Then*

$$\mathrm{Hess}(f_P)_Q(u, v) = 2s(\overrightarrow{PQ}, \Phi(u, v)) + 2s(u, v)$$

for all $u, v \in T_{X,Q}$.

Note that $s(\overrightarrow{PQ}, \Phi(u, v))$ is well-defined because $\overrightarrow{PQ}$ is orthogonal to $T_{X,Q}$ in the critical point Q.

Proof. We already know that

$$df_P(v) = 2s(\overrightarrow{Px}, v)$$

for all $v \in T_{X,x}$. We use local coordinates as in the last proof and consider $u = \partial_j$, $v = \partial_i$. We have

$$s(\overrightarrow{Px}, V_i) = (x_i - p_i) + \sum_{a=d+1}^{N} (g_a(x_1, \dots, x_d) - p_a) \frac{\partial g_a(x_1, \dots, x_d)}{\partial x_i}$$

and hence by applying ∂_j we compute

$$\partial_j s(\overrightarrow{Px}, V_i) = \delta_{ij} + \sum_{a=d+1}^{N} \frac{\partial g_a}{\partial x_j} \frac{\partial g_a}{\partial x_i} + \sum_{a=d+1}^{N} (g_a - p_a) \frac{\partial g_a}{\partial x_j \partial x_i}$$

The first two summands are $s(u, v)$ the last sum (after specialising to Q) are equal to $s(\overrightarrow{PQ}, \Phi(u, v))$ by the formula in the last proof. □

Proposition 10.2.6. *Let $X \subset \mathbb{C}^n$ be a smooth submanifold, $P \in \mathbb{C}^n$ such f_P is a Morse function. Assume that Q is a critical point. Then the Morse index is at most n.*

Proof. By the last computation we are left with the following set-up: We have d-dimensional $\mathbb{C}$-sub vector space $T \subset \mathbb{C}^n$, a $\mathbb{C}$-bilinear symmetric map $T \times T \to \mathbb{C}^n/T$. Fix $Q \in T^\perp$. We consider the $\mathbb{R}$-bilinear map $T \times T \to \mathbb{R}$

$$(u, v) \mapsto s(x, \Phi(u, v))) + s(u, v).$$

The form

$$H(u, v) = h(\Phi(u, v), x)$$

is $\mathbb{C}$-bilinear and symmetric, and

$$Q(u, v) = s(x, \Phi(u, v)) = s(\Phi(u, v), x) = \Re h(\Phi(u, v), x) = \Re H(u, v)$$

is $\mathbb{R}$-bilinear and symmetric. There is a decomposition $T = T^+ \oplus T^0 \oplus T^-$ (into $\mathbb{R}$-vector spaces) such that Q is positive definite on T^+, 0 on T^0, negative definite on T^-. These spaces are not unique, but their dimensions are. Let $t \in T^+$. Then it satisfies

$$Q(it, it) = H(it, it) = i^2 H(t, t) = -Q(t, t) < 0.$$

Hence Q is negative definite on iT^+, hence $\dim T^+ \leq \dim T^-$. By symmetry, we have $\dim T^+ = \dim T^-$. This implies $\dim T^- \leq d$. As s is positive definite on all of T, the sum $Q + s$ is positive definite on $T^+ \oplus T^0$. Hence the index of $Q + s$ is at most $\dim T^- \leq d$. □

This finishes the proof of Andreotti-Frankel.

10.3 Sard's theorem

We now prove Sard's Thereom (Theorem 6.1.3). For ease of reference, we recall the statement here.

Theorem 10.3.1 (Sard)**.** *Let $f : X \to \mathbb{R}^p$ be a smooth map of smooth manifolds. Let $\Sigma \subset X$ be the set of points where df_x is not surjective. Then $f(\Sigma)$ has measure 0 with respect to the Lebesgue measure.*

Proof. We follow [Mil97, pp. 16-19].

Let $n = \dim X$. As X is covered by countably many coordinate charts, it suffices to prove the theorem for $X \subset \mathbb{R}^n$ open. The argument is by induction on n. It is trivial for $n = 0$.

We consider the filtration

$$\Sigma \supset \Sigma_1 \supset \Sigma_2 \supset \dots$$

where Σ_i is the set of critical points where all partial derivatives of order $\leq i$ vanish. The proof has three steps:

Step 1 The image $f(\Sigma - \Sigma_1)$ has measure 0.

Step 2 The image $f(\Sigma_i - \Sigma_{i+1})$ has measure 0 for $i \geq 1$.

Step 3 The image $f(\Sigma_k)$ has measure 0 for k sufficiently large.

Step 1: We may assume $p \geq 2$ because $\Sigma = \Sigma_1$ for $p = 1$. We use Fubini: if $A \subset \mathbb{R}^p = \mathbb{R} \times \mathbb{R}^{p-1}$ is measurable and intersects all hyperplanes $\{c\} \times \mathbb{R}^{p-1}$ in a set of measure 0, then A has measure 0. For each $P \in \Sigma - \Sigma_1$, we will find an open neighbourhood $V \subset \mathbb{R}^n$ such that $f(V \cap \Sigma)$ has measure 0. Since $\Sigma_1 - \Sigma_1$ is covered by countably many of these, this is enough. Since $P \notin \Sigma_1$, there is a partial derivative, say $\partial f_1/\partial x_1$, that does not vanish in P. We consider

$$h : X \to \mathbb{R}^n, \quad x \mapsto (f_1(x), x_2, \dots, x_n).$$

Its Jacobian has full rank at P, hence there is a neighbourhood V of P mapped diffeomorphically to a neighbourhood V'. Consider the composition $g = f \circ h^{-1} : V' \to \mathbb{R}^p$. Its critical points are $\Sigma' = h(V \cap \Sigma)$, hence $g(\Sigma') = f(V \cap \Sigma)$. For each $(t, y_2, \dots, y_n) \in V'$ note that $g(t, y_2, \dots, y_n) \in \{t\} \times \mathbb{R}^{p-1} \subset \mathbb{R}^p$. Let

$$g^t : \{t\} \times \mathbb{R}^{n-1} \cap V' \to \{t\} \times \mathbb{R}^{p-1}$$

be the restriction of g. A point in the domain of g^t is critical for g^t if and only if it is critical for g. By the induction hypothesis its image has measure 0 in $\{t\} \times \mathbb{R}^{n-1}$. By the Fubini criterion this implies that $g(\Sigma') = f(\Sigma)$ has measure 0.

Step 2: Let $P \in \Sigma_k - \Sigma_{k+1}$. Hence there is some $(k+1)$st partial derivative $\partial^{k+1} f_r/\partial x_{s_1} \dots \partial x_{s_{k+1}}$ non-zero at P. Thus the function

$$\omega(x) = \frac{\partial^k f_r}{\partial x_{s_2} \dots \partial x_{s_{k+1}}}$$

vanishes at P, but $\partial \omega/\partial x_{s_1}$ does not. Without loss of generality, $s_1 = 1$. Consider $h : X \to \mathbb{R}^n$ given by

$$h(x) = (\omega(x), x_2, \dots, x_n)$$

carries some neighbourhood V of P diffeomorphically onto $V' \subset \mathbb{R}^n$. It maps $\Sigma_k \cap V$ to $\{0\} \times \mathbb{R}^{n-1}$. Again consider

$$g = f \circ h^{-1} : V' \to \mathbb{R}^p.$$

Let $\bar{g}$ be the restriction of g to $\{0\} \times \mathbb{R}^{n-1}$. By induction, the set of critical values of $\bar{g}$ has measure zero in $\mathbb{R}^n$. All point in $h(\Sigma_k \cap V)$ are critical points of $\bar{g}$ (since all partial derivatives to order $\leq k$ vanish), therefore

$$\bar{g}h(\Sigma_k \cap V) = f(\Sigma_k \cap V)$$

has measure 0. As in the first step this suffices to show that $f(\Sigma_k - \Sigma_{k+1})$ has measure 0.

Step 3: Let $I^n \subset X$ be a cube with edge δ, $k+1 > n/p$. We will prove that $f(\Sigma_k \cap I^n)$ has measure 0. We use the Taylor expansion and the definition of Σ_k. For $P \in \Sigma_k \cap I^n$

$$f(P+h) = f(P) + R(P,h)$$

where

$$\|R(P,h)\| \leq c\|h\|^{k+1}$$

for $P+h \in I^n$. Here c is a constant that depends only on f and I^n. We subdivide I^n into r^n cubes of edge δ/r. Let I be a cube in the subdivision containing a point $P \in \Sigma_k$. Any point in I can be written as $P+h$ with

$$\|h\| \leq \sqrt{n}(\delta/r)$$

Hence $f(I)$ lies in a cube of edge a/r^{k+1} centered at $f(P)$ where $a = 2c(\sqrt{n}\delta)^{k+1}$ is constant. Hence $f(\Sigma_k \cap I^n)$ is contained in a union of at most r^n cubes having total volume

$$V \leq r^n(1/r^{k+1})^p = a^p r^{n-(k+1)p}.$$

By our assumption on k, the exponent $r-(k+1)p$ is negative, hence the right hand side tends to 0 for $r \to \infty$. This means that $f(\Sigma_k \cap I^n)$ has measure 0.

This completes the proof. □

References

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