Topology of algebraic varieties

Brad Drew and Annette Huber

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1 Topological spaces

Motivation. Our primary objects of interest will be "spaces" contained in Euclidean spaces $\mathbf{C}^n$, e.g., the subset

$$\{(x,y) \in \mathbf{C}^n \mid y^2 = x^3 + x + 1\} \subseteq \mathbf{C}^2.$$

Each such space inherits a notion of distance from $\mathbf{C}^n = \mathbf{R}^{2n}$. In studying these spaces, however, we will be led to consider spaces that are not contained in a Euclidean space in an obvious way, such as the tangent space $\mathrm{T}X$ of a manifold X. We therefore begin by introducing a notion of space that is sufficiently intrinsic and robust that we need not worry about constructing embeddings into Euclidean spaces.

1.1 Topological spaces and continuous maps

Open and closed subsets of Euclidean spaces

Definition 1.1.1. Let $\mathbf{E} = \mathbf{R}^n$ or $\mathbf{E} = \mathbf{C}^n$, let $z = (z_1, \dots, z_n) \in \mathbf{E}$, and let $r \in \mathbf{R}_{\geq 0}$.

(1) We let $|\cdot| \colon \mathbf{E} \to \mathbf{R}$ denote the *Euclidean norm*, given by

$$|z| \coloneqq \sqrt{z_1\bar{z}_1 + \cdots + z_n\bar{z}_n} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

(2) The *open disk* $\mathbf{D}(z,r)$ *of radius* r *centered at* z is the subset

$$\mathbf{D}(z,r) \coloneqq \{w \in \mathbf{E} \mid |w-z| < r\} \subseteq \mathbf{E}.$$

(3) The *closed disk* $\bar{\mathbf{D}}(z,r)$ *of radius* r *centered at* z is the subset

$$\bar{\mathbf{D}}(z,r) \coloneqq \{w \in \mathbf{E} \mid |w-z| \leq r\} \subseteq \mathbf{E}.$$

(4) The *boundary* $\partial\mathbf{D}(z,r)$ *of the disk of radius* r *centered at* z is the subset

$$\partial\mathbf{D}(z,r) \coloneqq \{w \in \mathbf{E} \mid |w-z| = r\} = \bar{\mathbf{D}}(z,r) - \mathbf{D}(z,r) \subseteq \mathbf{E}.$$

(5) The *punctured open disk of radius r centered at z* is the subset

$$\mathbf{D}^*(z,r) \coloneqq \{w \in \mathbf{E} \mid 0 < |w-z| < r\} = \mathbf{D}(z,r) - \{z\} \subseteq \mathbf{E}.$$

(6) The *punctured closed disk of radius r centered at z* is the subset

$$\bar{\mathbf{D}}^*(z,r) \coloneqq \{w \in \mathbf{E} \mid 0 < |w-z| \leq r\} = \bar{\mathbf{D}}(z,r) - \{z\} \subseteq \mathbf{E}.$$

(7) A subset $\subseteq \mathbf{E}$ is an *open polydisk* if it is of the form

$$\mathbf{D}(z_1,r_1) \times \cdots \times \mathbf{D}(z_n,r_n)$$

for some $r_1,\ldots,r_n \in \mathbf{R}_{\geq 0}$. We define closed polydisks similarly.

Definition 1.1.2. Let $\mathbf{E} = \mathbf{R}^n$ or $\mathbf{E} = \mathbf{C}^n$ and let $X \subseteq \mathbf{E}$.

(1) We say that X is *open in* $\mathbf{E}$ if, for each $z \in X$, there exists $\varepsilon \in \mathbf{R}_{>0}$ such that $\mathbf{D}(z,r) \subseteq X$.

(2) We say that X is *closed in* $\mathbf{E}$ if the complementary subset $\mathbf{E} - X$ is open.

(3) More generally, if $X \subseteq Y \subseteq \mathbf{E}$, then we say that X is *open in* Y or *closed in* Y, respectively, if there exists an open or closed subset $X' \subseteq \mathbf{E}$ such that $X = X' \cap Y$.

Exercise 1.1.3. Subsets are not doors. Given examples of subsets $X \subseteq \mathbf{R}^n$ with the following properties:

(1) X is neither open nor closed;
(2) X is open but not closed;
(3) X is closed but not open;
(4) X is both open and closed.

Exercise 1.1.4. Let $\mathbf{E} = \mathbf{R}^n$ or $\mathbf{C}^n$, let $U \subseteq \mathbf{E}$ be an open subset, and let $x = (x_1,\ldots,x_n) \in U$. There exists $(r_1,\ldots,r_n) \in \mathbf{R}^n_{\geq 0}$ such that the open polydisk

$$\mathbf{D}(x_1,r_1) \times \cdots \times \mathbf{D}(x_n,r_n) \tag{1.1.4.a}$$

is contained in U.

Definition 1.1.5. Let $\mathbf{E} = \mathbf{R}^n$ or $\mathbf{C}^n$. A subset $X \subseteq \mathbf{E}$ is *convex* if, for each pair of points $x, y \in \mathbf{E}$, the line segment L joining them in $\mathbf{E}$ is contained in X.

Exercise 1.1.6. Let $\mathbf{E} = \mathbf{R}^n$ or $\mathbf{C}^n$, let $(x_1,\ldots,x_n) \in \mathbf{E}$, and let $(r_1,\ldots,r_n) \in \mathbf{R}^n_{\geq 0}$. The polydisk (1.1.4.a) is convex.

Topological spaces

Motivation. By virtue of this definition of open subsets of $\mathbf{R}^n$, two points $x, y \in \mathbf{R}^n$ are "close" to one another, i.e., the distance $|x-y|$ between them is "small", precisely when x and y belong to "many" of the same open subsets. It turns out that using the notion of open subsets to characterize proximity has certain advantages over that of a distance function. We will therefore use an axiomatization of the properties of open subsets as the basis for our most basic notion of "space", i.e., that of a "topological space". We will later need to consider a more refined notion of space—that of a "smooth manifold"—in order to gain access to the formalism of differential and integral calculus.

Topological spaces and subspaces

Definition 1.1.7. Let X be a set. A *topology τ on X* is a set of subsets of X satisfying the following conditions:

(1) $\varnothing \in \tau$ and $X \in \tau$;
(2) if I is a set and $U_i \in \tau$ for each $i \in I$, then $\bigcup_{i \in I} U_i \in \tau$; and
(3) if J is a finite set and $U_j \in \tau$ for each $j \in J$, then $\bigcap_{j \in J} U_j \in \tau$.

Definition 1.1.8. A *topological space* is a pair (X, τ) consisting of a set X and a topology τ on X.

Definition 1.1.9. Let (X, τ) be a topological space.

(1) A subset $U \subseteq X$ is *open with respect to* τ if $U \in \tau$.
(2) A subset $Z \subseteq X$ is a *closed with respect to* τ if the complement $X - Z$ is open.

Exercise 1.1.10. Let X be a set. Show that the datum of a family of open sets for a topology τ on X is equivalent to that of a family σ satisfying the following conditions:

(1) $\varnothing \in \sigma$ and $X \in \tau$;
(2) if I is a set and $Z_i \in \sigma$ for each $i \in I$, then $\bigcap_{i \in I} Z_i \in \sigma$; and
(3) if J is a finite set and $Z_j \in \sigma$ for each $j \in J$, then $\bigcup_{j \in J} Z_j \in \sigma$.

Here, σ corresponds to the set of closed subsets of the topology τ.

Neighborhoods, closures, interiors, and boundaries

Definition 1.1.11. Let X be a topological space and let $V \subseteq W \subseteq X$. We say that W is a *neighborhood of* V *in* X if there exists an open subset U of X such that $Y \subseteq U \subseteq W$. If $x \in X$ and $V = \{x\}$, then we say that W is a *neighborhood of* x *in* X.

Definition 1.1.12. Let X be a toplogical space and let $A \subseteq X$ be a subset.

(1) The *closure* $\bar{A}$ *of* A *in* X is the smallest closed subset of X containing A, i.e., the intersection of all closed subsets $Z \subseteq X$ such that $A \subseteq Z$.

(2) The *interior* A° *of* A *in* X is the largest open subset of X contained in A, i.e., the union of all open subsets $U \subseteq X$ such that $U \subseteq A$.

(3) The *boundary* ∂A *of* A *in* X is the relative complement $\bar{A} - A^\circ$. Note that ∂A is closed in X.

Exercise 1.1.13. Let X be a topological space. A subset $A \subseteq X$ is open (resp. closed) if and only if $A = A^\circ$ (resp. $A = \bar{A}$).

Proposition 1.1.14. *Let X be a toplogical space, let $A \subseteq X$ be a subset, and let $x \in X$.*

(1) *The point x belongs to $X - \bar{A}$ if and only if there exists an open neighborhood $x \in U \subseteq X$ such that $U \cap A = \varnothing$.*

(2) *The point x belongs to A° if and only if there exists an open neighborhood $x \in U \subseteq X$ such that $U \cap (X - A) \neq \varnothing$.*

(3) *The point x belongs to ∂A if and only if, for each open neighborhood $x \in U \subseteq X$, neither of the intersections $U \cap A$ nor $U \cap (X - A)$ is empty.*

Proof. Consider Claim (1). Suppose that $U \cap A = \varnothing$ for some open neighborhood U of x. We then have $A \subseteq X - U$. As $X - U$ is closed, and $\bar{A}$ is the smallest closed subset of X containing A, we have $\bar{A} \subseteq X - U$, or, equivalently, $x \in U \subseteq X - \bar{A}$. For the converse, consider the open neighborhood $U \coloneqq X - \bar{A}$.

Consider Claim (2). Suppose that $U \cap (X - A) = \varnothing$ for some open neighborhood U of x. We then have $U \subseteq A$. As A° is the largest open subset of X contained in A, we have $x \in U \subseteq A^\circ$. For the converse, consider the open neighborhood $U \coloneqq A^\circ$.

Consider Claim (3). Combine (1) and (2), recalling that $\partial A = \bar{A} - A^\circ = (X - A^\circ) \cap (X - (X - \bar{A}))$. $\square$

Examples of topological spaces

Example 1.1.15. Let $n \in \mathbf{Z}_{\geq 0}$. The open subsets of $\mathbf{R}^n$ as defined in Definition 1.1.2 form a topology on $\mathbf{R}^n$, which we refer to as the *standard* or *Euclidean topology.*

Example 1.1.16. Let $n \in \mathbf{Z}_{\geq 0}$. The *unit n-sphere* $\mathbf{S}^n$ is the subspace

$$\mathbf{S}^n := \{x \in \mathbf{R}^{n+1} \mid |x| = 1\} \subseteq \mathbf{R}^{n+1}.$$

Example 1.1.17. Let X be a set.

(1) The *discrete topology on* X is the topology in which each subset of X is declared to be open.

(2) The *indiscrete topology on* X is the topology in which $\varnothing$ and X are both open, and no other subsets are open.

Example 1.1.18. The *cofinite topology on* $\mathbf{R}$ is the topology in which a subset $U \subseteq \mathbf{R}$ is declared to be open if its complement $\mathbf{R} - U$ is a finite set.

Remark 1.1.19. If $U \subseteq \mathbf{R}$ is open in the cofinite topology, then it is open in the standard topology. The converse is not true. We say that the standard topology is *finer than* the cofinite topology. The indiscrete topology is the coarsest topology, and the discrete topology is the finest. In general, given two topologies τ and τ' on a set X, one is not necessarily finer than the other: refinement is only a partial ordering on the set of topologies.

Example 1.1.20. Let $n \in \mathbf{Z}_{\geq 0}$.

(1) Let S be a set of polynomials with complex coefficients in n variables. Their *vanishing locus* $\mathbf{V}(S) \subseteq \mathbf{C}^n$ is the set

$$\mathbf{V}(S) := \{z \in \mathbf{C}^n \mid \forall f \in S\,[f(z) = 0]\},$$

of their common zeros.

(2) A subset $X \subseteq \mathbf{C}^n$ is *algebraic* if it is of the form $X = \mathbf{V}(S)$ for some family of polynomials S.

(3) If $(S_i)_{i \in I}$ is a set of families of polynomials, then the vanishing locus of the union is the intersection of the sets $\mathbf{V}(S_i)$ with $i \in I$:

$$\mathbf{V}\Big(\bigcup_{i \in I} S_i\Big) = \bigcap_{i \in I} \mathbf{V}(S_i).$$

In other words, arbitrary intersections of algebraic sets are algebraic. If S and T are two families of polynomials in n variables, let $S \cdot T$ denote the set of polynomials given by

$$S \cdot T := \{f \cdot g \mid f \in S,\, g \in T\}.$$

With this notation, we have

$$\mathbf{V}(S) \cup \mathbf{V}(T) = \mathbf{V}(S \cdot T),$$

i.e., finite unions of algebraic sets are algebraic. As $\varnothing = \mathbf{V}(\{1\})$ and $\mathbf{C}^n = \mathbf{V}(\{0\})$, it follows from Exercise 1.1.10 that the algebraic subsets are the closed subsets for a unique topology on $\mathbf{C}^n$, which we refer to as the *Zariski topology.*

(4) More generally, if $X \subseteq \mathbf{C}^n$ is any subset, the *Zariski topology on* X is the subspace topology on X associated with the Zariski topology on $\mathbf{C}^n$.

Exercise 1.1.21. Identify all possible topologies on the following sets:

(1) $\{0\}$

(2) $\{0, 1\}$

(3) $\{0, 1, 2\}$

Subspaces

Definition 1.1.22. Let (X, τ) be a topological space and let $Y \subseteq X$ be a subset. The *subspace topology on* Y is the topology on Y whose open sets are of the form $U \cap Y$ with $U \in \tau$.

Exercise 1.1.23. Let (X, τ) be a topological space and let Y be a subset. Prove the following assertions.

(1) If $Y \subseteq X$ is open, then $U \subseteq Y$ is open in the subspace topology if and only if U is an open subset of X.

(2) A subset $Z \subseteq Y$ is closed in the subspace topology if and only if there exists a closed subset $Z' \subseteq X$ such that $Z = Z' \cap Y$.

Limit points

Definition 1.1.24. Let X be a topological space, let $A \subseteq X$ be a subset, and let $x \in X$. We say that x is a *limit point of* A if, for each neighborhood $x \in N \subseteq X$, $N \cap A \neq \varnothing$.

Proposition 1.1.25. *Let X be a topological space. The subset $A \subseteq X$ is closed if and only if A contains each of its limit points.*

Proof. Suppose that A is closed. If $x \in X - A$, then $X - A$ is an open neighborhood of x not meeting A, so x is not a limit point of A. Conversely, suppose that A contains each of its limit points. If $x \in X - A$, then x is not a limit point of A, so there exists a neighborhood of x contained in $X - A$. This neighborhood contains an open neighborhood by definition, so $X - A$ is open. □

Continuous maps

Definition 1.1.26. Let X and Y be topological spaces and $f : X \to Y$ a function. We say that f is:

(1) *continuous* if $f^{-1}(V) \subseteq X$ is open for each open set $V \subseteq Y$;

(2) *open* if $f(U) \subseteq Y$ is open for each open $V \subseteq X$;

(3) *closed* if $f(Z) \subseteq Y$ is closed for each closed $Z \subseteq X$; and

(4) a *homeomorphism* if it is continuous and it admits a continuous inverse, i.e., a continuous function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Remark 1.1.27. If there exists a homeomorphism $f : X \to Y$, then X and Y are indistinguishable in terms of the topological properties that they satisfy. For the reader familiar with the categorical notion of an isomorphism, homeomorphisms are precisely the isomorphisms in the category of topological spaces and continuous maps.

Proposition 1.1.28. *Let X, Y and Z be topological spaces.*

(1) *The identity map* $\mathrm{id}_X : X \to X$ *is a homeomorphism and, in particular, it is continuous.*

(2) *If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then the composite $g \circ f : X \to Z$ is continuous.*

(3) *If X is a subspace of Y and $i : X \hookrightarrow Y$ is the inclusion map, then i is continuous.*

(4) *If X is an open (resp. closed) subspace of Y and $i : X \hookrightarrow Y$ is the inclusion map, then i is open (resp. closed).*

Proof. By definition, $\mathrm{id}_X(x) = x$ for each $x \in X$, so $\mathrm{id}_X = \mathrm{id}_X^{-1}$. Thus, Claim (1) is the vacuous assertion that $U \subseteq X$ is open if and only if $U \subseteq X$ is open.

Consider Claim (2). Let $U \subseteq Z$ be open. We have $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Continuity of f implies that $f^{-1}(U)$ is open, and the claim now follows from continuity of g.

Claim (3) is essentially the definition of the subspace topology (Definition 1.1.22).

Consider Claim (4). Suppose that X is an open (resp. closed) subspace of Y and W is an open (resp. closed) subspace of X. By definition of the subspace topology, $i(W) = W = W' \cap X$ for some open (resp. closed) subset $W' \subseteq Y$. As the intersection of two open (resp. closed) subsets is another such, the claim follows. □

Example 1.1.29. Let $n \in \mathbf{Z}_{>0}$, let $U \subseteq \mathbf{R}^n$ be an open subset, and let $f \colon U \to \mathbf{R}$ be a function. The following conditions are equivalent:

(1) f is continuous;

(2) for each $\varepsilon \in \mathbf{R}_{>0}$ and each $x \in U$, there exists $\delta \in \mathbf{R}_{>0}$ such that, for each $y \in U$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Example 1.1.30. Let $X \subseteq \mathbf{R}^2$ be the subset consisting of the points (x, y) such that $y = x^2$. The projection $\pi \colon (x, y) \mapsto x \colon X \to \mathbf{R}$ is a homeomorphism with inverse $\iota \colon x \mapsto (x, x^2) \colon \mathbf{R} \to X$.

Exercise 1.1.31. Let $f \colon X \to Y$ be a continuous map of topological spaces that is bijective on underlying sets. Show that the following conditions are equivalent:

(1) f is a homeomorphism;

(2) f is open; and

(3) f is closed.

Exercise 1.1.32. Provide an example of a continuous bijection that is not a homeomorphism.

Exercise 1.1.33. Construct a homeomorphism $f \colon \mathbf{D}(x, r) \to \mathbf{R}$ for $n \in \mathbf{Z}_{>0}$, $x \in \mathbf{R}^n$, and $r \in \mathbf{R}_{>0}$.

1.2 Product spaces, gluing, and quotient spaces

Product topology

Proposition 1.2.1. *Let X and Y be topological spaces. There is a topological space $X \times Y$ equipped with continuous maps $p : X \times Y \to X$ and $q : X \times Y \to Y$ such that, for each topological space W equipped with continuous maps $f : W \to X$ and $g : W \to Y$, there exists a unique continuous map $h : W \to X \times Y$ such that $f = p \circ h$ and $g = q \circ h$. Moreover, the data $(X \times Y, p, q)$ is unique up to unique homeomorphism.*

Proof. The set underlying $X \times Y$ is the Cartesian product of the sets underlying X and Y, i.e., it is the set of ordered pairs (x, y) such that $x \in X$ and $y \in Y$. We define a topology on this set by declaring that a subset $W \subseteq X \times Y$ is open if it is a union of subsets of the form $U \times V$ with $U \subseteq X$ and $V \subseteq Y$ open subsets. This is indeed a topology: $\varnothing = \varnothing \times \varnothing$ and $X \times Y$ are both open in this sense, open subsets are stable under arbitrary unions by fiat, and the equalities

$$\Big(\bigcup_{\alpha \in A} (U_\alpha \times V_\alpha)\Big) \cap \Big(\bigcup_{\alpha \in A} (U'_\beta \times V'_\beta)\Big) = \bigcup_{\alpha \in A} \bigcup_{\beta \in B} ((U_\alpha \times V_\alpha) \cap (U'_\beta \times V'_\beta))$$
$$= \bigcup_{\alpha \in A} \bigcup_{\beta \in B} ((U_\alpha \cap U'_\beta) \times (V_\alpha \cap V'_\beta))$$

show that they are also stable under finite intersections.

Let W be a topological space and let $f \colon W \to X$ and $g \colon W \to Y$ be continuous maps. The function $(f, g) \colon W \to X \times Y$ given by $(f, g)(w) = (f(w), g(w))$ is the unique map of sets h such that $f = p \circ h$ and $g = q \circ h$. It remains to show that (f, g) is continuous. We may write each open subset of $X \times Y$ in the form $\bigcup_{\alpha \in A} (U_\alpha \times V_\alpha)$ with U_α and V_α open subsets of X and Y, respectively, for each $\alpha \in A$, and we have

$$(f, g)^{-1}\Big(\bigcup_{\alpha \in A} (U_\alpha \times V_\alpha)\Big) = \bigcup_{\alpha \in A} (f, g)^{-1}(U_\alpha \times V_\alpha) = \bigcup_{\alpha \in A} (f^{-1}(U_\alpha) \cap g^{-1}(V_\alpha)),$$

which is open in W by continuity of f and g.

The last assertion follows from the uniqueness of the map (f, g) we have just constructed. Indeed, if $(P, p' \colon P \to X, q' \colon P \to Y)$ is another triple satisfying the same conditions as $(X \times Y, p, q)$, then there exist unique continuous maps $h \colon X \times Y \to P$ and $k \colon P \to X \times Y$ compatible with p', q', p, and q. Applying the same uniqueness condition to the composites $h \circ k$ and $k \circ h$, we find that $h \circ k$ and $k \circ h$ are identity morphisms, so h and k are mutually inverse homeomorphisms. □

Definition 1.2.2. We refer to the topological space W of Proposition 1.2.1 as the *product of X and Y*, and we refer to the maps p and q as the *projections*.

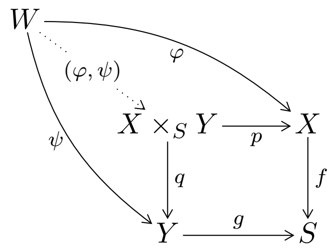
Example 1.2.3. For each $m, n \in \mathbf{Z}_{\geq 0}$, the projections $\mathbf{R}^{m+n} \to \mathbf{R}^m$ and $\mathbf{R}^{m+n} \to \mathbf{R}^n$ induce a unique continuous map $\mathbf{R}^{m+n} \to \mathbf{R}^m \times \mathbf{R}^n$. This map is a homeomorphism.

Definition 1.2.4. Let $f \colon X \to S$ and $g \colon Y \to S$ be continuous maps of topological spaces. The *fiber product $X \times_S Y$ of X and Y over S* is the subspace

$$X \times_S Y \coloneqq \{(x, t) \in X \times Y \mid f(x) = g(t)\} \subseteq X \times Y.$$

We refer to the composites $p \colon X \times_S Y \hookrightarrow X \times Y \to X$ and $q \colon X \times_S Y \hookrightarrow X \times Y \to Y$ of the inclusion with the projection maps on $X \times Y$ as the *projection maps*.

Exercise 1.2.5. With the notation and hypotheses of Definition 1.2.4, show that, for each commutative square of solid arrows



consisting of continuous maps of topological spaces, there exists a unique continuous map $(\varphi, \psi) \colon W \to X \times_S Y$ such that $\varphi = p \circ (\varphi, \psi)$ and $\psi = q \circ (\varphi, \psi)$. This is the *universal property of the fiber product*.

Quotient topology

Proposition 1.2.6. *Let X be a topological space, and let R be an equivalence relation on the set underlying X. There is a topological space X/R equipped with a surjective continuous map $p : X \to X/R$ such that, for each continuous map of topological spaces $f : X \to Y$ such that $f(x) = f(x')$ for each $(x, x') \in R$, there exists a unique continuous map $\bar{f} : X/R \to Y$ such that $f = \bar{f} \circ p$. Moreover, the data $(X/R, p)$ is unique up to unique homeomorphism.*

Proof. The set underlying X/R is the set of equivalence classes

$$[x] \coloneqq \{y \in X \mid (x, y) \in R\}$$

for the relation R. There is a natural surjection $\pi x \mapsto [x] \colon : X \to X/R$. We declare a subset $U \subseteq X/R$ to be open if its preimage $\pi^{-1}(U)$ is open in X. These are the open subsets for a topology on X/R: $\varnothing$ and X are open in this sense, and preimages preserve unions and intersections, so open subsets in this sense are stable under arbitrary unions and finite intersections by virtue of the analogous properties for the given topology on X. By construction, π is continuous with respect to this topology.

Let $f \colon X \to Y$ be a continuous map that is constant on the equivalence classes of the relation R. As maps of sets, f factors uniquely in the form $\bar{f} \circ \pi$: $\bar{f}([x]) = f(x)$ for each $x \in X$. If $U \subseteq Y$

is open, then $f^{-1}(U) = \pi^{-1}(\bar{f}^{-1}(U)) \subseteq X$ is open by continuity of f and, by definition of the topology on X/R, this means that $\bar{f}^{-1}(U)$ is open in X/R, so $\bar{f}$ is continuous.

Suppose that $(Q, \varphi\colon X \to Q)$ is another pair satisfying the same conditions as $(X/R, \pi)$. These conditions provide us with unique continuous maps $\bar{\pi}\colon Q \to X/R$ and $\bar{\varphi}\colon X/R \to Q$. By uniqueness, the composites $\bar{\pi} \circ \bar{\varphi}$ and $\bar{\varphi} \circ \bar{\pi}$ must be the identity maps, so $\bar{\pi}$ and $\bar{\varphi}$ are mutually inverse homeomorphisms. ☐

Definition 1.2.7. We refer to the space X/R of Proposition 1.2.6 as the *quotient space of X by R* and we refer to p as the *quotient map.*

Example 1.2.8. $\mathbf{S}^2$, $\mathbf{T}^2$, Möbius strip, Klein bottle, $\mathbf{RP}^2$

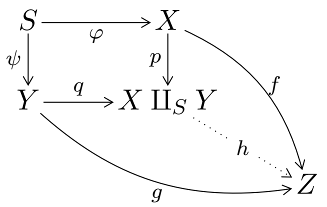
Example 1.2.9 (Complex projective space)**.** Let $n \in \mathbf{Z}_{\geq 0}$ and consider the subspace $\mathbf{C}^{n+1} - \{0\} \subseteq \mathbf{C}^{n+1}$. We define an equivalence relation on $\mathbf{C}^{n+1} - \{0\}$ by declaring that $x \simeq y$ if and only if there exists $\lambda \in \mathbf{C}$ such that $x = \lambda y$, i.e., if and only if x and y span the same 1-dimensional subspace. The quotient space of $\mathbf{C}^{n+1} - \{0\}$ by this equivalence relation is the *n-dimensional complex projective space* $\mathbf{CP}^n$. It will be a fundamental example in the sequel.

Gluing construction

Proposition 1.2.10. *Consider the following data and hypotheses:*

- *S, X and Y are topological spaces; and*
- *$\varphi : S \to X$ and $\psi : S \to Y$ are continuous maps.*

There exists a topological space $X \amalg_S Y$ equipped with continuous maps $p : X \to X \amalg_S Y$ and $q : Y \to X \amalg_S Y$, such that for each commutative diagram of solid arrows



there exists a unique continuous map h rendering the diagram commutative. Moreover, the triple $(X \amalg_S Y, p, q)$ is unique up to unique homeomorphism.

Proof. Consider the disjoint unions $X \amalg Y$ and $X \amalg S \amalg Y$. The morphisms id_X, φ, ψ, and id_Y induce a unique continuous map $\sigma\colon X \amalg S \amalg Y \to X \amalg Y$. Define an equivalence relation R on $X \amalg Y$ by declaring that $(x, y) \in R$ if and only if $\sigma(x) = \sigma(y)$. Equip $X \amalg_S Y \coloneqq (X \amalg Y)/R$ with the quotient topology of Proposition 1.2.6, and let p and q denote the composites

$$X \hookrightarrow X \amalg Y \twoheadrightarrow X \amalg_S Y \qquad \text{and} \qquad Y \hookrightarrow X \amalg Y \twoheadrightarrow X \amalg_S Y,$$

which are continuous as composites of continuous maps are continuous.

Consider continuous maps $f\colon X \to Z$ and $g\colon Y \to Z$ such that $f \circ \varphi = g \circ \psi$. The maps φ and ψ induce a unique continuous map $\tilde{h}\colon X \amalg Y \to Z$. By construction, $\tilde{h}$ is constant on equivalence classes for R, so it factors uniquely through a continuous map $X \amalg_S Y \to Z$ by Proposition 1.2.6.

The proof of last assertion is the same as in Proposition 1.2.1 and Proposition 1.2.6. ☐

Definition 1.2.11. We refer to the topological space $X \amalg_S Y$ of Proposition 1.2.10 as the *space obtained by gluing X and Y along S.*

Example 1.2.12. $\mathbf{RP}^n$, $\mathbf{CP}^n$

1.3 Connected, Hausdorff, and quasi-compact spaces

Connected spaces

Definition 1.3.1. The topological space X is *connected* if it is not the disjoint union of two nonempty open subsets: for each pair of open subsets $U, V \subseteq X$, if $U \cap V = \varnothing$ and $U \cup V = X$, then $U = \varnothing$ or $V = \varnothing$.

Proposition 1.3.2. *With respect to the Euclidean topology,* $\mathbf{R}$ *is connected, as is the subspace* $[a, b]$ *for each* $a, b \in \mathbf{R}$.

Proof. The empty set and the singleton are both connected, so we may assume without loss of generality that $a < b$. Let X denote either $\mathbf{R}$ or $[a, b]$. Let $U, V \subseteq X$ be disjoint open subsets such that $U \cup V = X$. Let $x \in U$ and $y \in V$. Without loss of generality, we may assume that $x < y$. Let z denote the infimum of $V \cap [x, y]$. In particular, $[x, z) \cap V = \varnothing$. As $U = -V$ is open, V is closed in X, so it must contain its limit point z. As U is also closed, by openness of V, and z is a limit point of $[x, z) \subseteq U$, we must also have $z \in U$, contradicting the hypothesis that $U \cap V = \varnothing$. ☐

Example 1.3.3. $\mathbf{D}(x, \varepsilon)$, $\{x = 0\} \cup \{y = 0\}$, $\{x = 0\} \cup \{x = 1\}$

Proposition 1.3.4. *A topological space* X *is connected if and only if* $\varnothing$ *and* X *are the only subsets of* X *that are simultaneously open and closed.*

Proof. Given a pair of nonempty, disjoint open subsets $U, V \subseteq X$ witnessing the nonconnectedness of X, U and V are both closed, as their respective complements V and U are both open. Conversely, if $\varnothing \neq U \subsetneq X$ is open and closed, then $\varnothing \neq V := X - U \subsetneq X$ is also open, disjoint from U, and complementary to U. ☐

Proposition 1.3.5. *If* $f\colon X \to Y$ *is a continuous map and* X *is connected, then the subspace* $f(X) \subseteq Y$ *is connected.*

Proof. Factoring f as the composite $X \to f(X) \hookrightarrow Y$, we may assume without loss of generality that $f(X) = Y$. Let U and V be disjoint open subsets of Y such that $U \cup V = Y$. In particular, U and V are also closed. The preimages $f^{-1}(U)$ and $f^{-1}(V)$ are open and closed, and their union is X. Their intersection must also be empty, as it is the preimage of $U \cap V = \varnothing$. ☐

Example 1.3.6. $\mathbf{S}^1$ is connected as the continuous image of $f\colon \theta \mapsto \exp(\mathrm{i}\theta)\colon \mathbf{R} \to \mathbf{C}$

Example 1.3.7. Let $x \in \mathbf{R}^2$, let $r \in \mathbf{R}_{>0}$, and let $D := \mathbf{D}(0, r)$. We claim that the subspace $X = \mathbf{R}^2 - D \subseteq \mathbf{R}^2$ is connected.

Suppose for contradiction that $X = U \cup V$ with $U, V \subseteq X$ open, and that $U \cap V = \varnothing$. First, note that U and V are closed in $\mathbf{R}^2$: they are closed in $\mathbf{R}^2 - D$, and $\mathbf{R}^2 - D$ is closed in $\mathbf{R}^2$.

We now observe that $U \cap \partial D \neq \varnothing$. Suppose for contradiction that $\partial D \cap U = \varnothing$. In this case, we have $\partial D \subseteq V$ and, hence, U and $V \cup D = V \cup \bar{D}$ are disjoint closed subsets of $\mathbf{R}^2$ and $U \cup (V \cup \partial D) = \mathbf{R}^2$, which contradicts the fact that $\mathbf{R}^2$ is connected. The same argument shows that $V \cap \partial D \neq \varnothing$.

It follows that $(U \cap \partial D)$ and $(V \cap \partial D)$ are disjoint closed subsets whose union is ∂D. As ∂D is homeomorphic to $\mathbf{S}^1$, it is connected, and we have the required contradiction.

Lemma 1.3.8. *Let* X *be a topological space and let* $\{Y_\alpha\}_{\alpha \in A}$ *be a family of connected subspaces of* X*. If there exists* $\alpha_0 \in A$ *such that* $Y_{\alpha_0} \cap Y_\alpha \neq \varnothing$ *for each* $\alpha \in A$*, then* $Y := \bigcup_{\alpha \in A} Y_\alpha$ *is connected.*

Proof. Let U and V be disjoint, open subsets of Y such that $Y = U \cup V$. For each $\alpha \in A$, the hypothesis that Y_α is connected implies that $Y_\alpha \subseteq U$ or $Y_\alpha \subseteq V$. Indeed, $U \cap Y_\alpha$ and $V \cap Y_\alpha$ are disjoint open subsets of Y_α whose union in Y_α, so one of them must be empty. Without loss of generality, we may assume that $Y_{\alpha_0} \subseteq U$. If $Y_\alpha \subseteq U$ for each $\alpha \in A$, then $Y \subseteq U$. If not, there exists $\alpha \in A$ such that $Y_\alpha \subseteq V$. In that case, $Y_{\alpha_0} \cap Y_\alpha \subseteq U \cap V = \varnothing$. □

Example 1.3.9. For each $n \in \mathbf{Z}_{>1}$, the subspace $\mathbf{R}^n - \{0\} \subseteq \mathbf{R}^n$ is connected. Let L be a line in $\mathbf{R}^n$ not passing through the origin and let $\{L_\alpha\}_{\alpha \in A}$ denote the set of all lines in $\mathbf{R}^n$ not passing through the origin and intersecting L. In particular, there exists $\alpha_0 \in A$ such that $L = L_{\alpha_0}$. By Proposition 1.3.2, L_α is connected for each α. By Lemma 1.3.8, it suffices to show that the union $\bigcup_{\alpha \in a} L_\alpha$ is $\mathbf{R}^n - \{0\}$. Let $x \in \mathbf{R}^n - \{0\}$. If $x \in L$, then there is nothing to show. If $x \notin L$, then x and L determine a plane in $\mathbf{R}^n$. As 0 lies on at most one line in this plane, we may choose a line meeting x and L not containing 0. This line is L_α for some α, so $x \in L_\alpha$.

Example 1.3.10. The space $\mathbf{CP}^n$ is connected for each $n \in \mathbf{Z}_{\geq 0}$.

Proof. If $n = 0$, then $\mathbf{CP}^0$ is a singleton, hence connected. Suppose that $n > 0$. The space $\mathbf{C}^{n+1} - \{0\}$ is connected by Example 1.3.9, and the quotient map $\mathbf{C}^{n+1} - \{0\} \to \mathbf{CP}^n$ is surjective and continuous by definition, so $\mathbf{CP}^n$ is connected by Proposition 1.3.5. □

Connected components

Definition 1.3.11. Let X be a topological space. A *connected component of* X is a maximal connected subspace, that is, a connected subspace $Y \subseteq X$ such that, for each connected subspace $Y' \subseteq X$, if $Y \subseteq Y'$, then $Y = Y'$.

Proposition 1.3.12. *Let X be a topological space and let $\{Y_\alpha\}_{\alpha \in A}$ be a family of connected subspaces of X. If the intersection $\bigcap_{\alpha \in A} Y_\alpha$ is nonempty, then the union $Y = \bigcup_{\alpha \in A} Y_\alpha$ is connected.*

Proof. Let x belong to the intersection and let U and V be disjoint open subsets of Y whose union is Y. Without loss of generality, suppose that $x \in U$. For each $\alpha \in A$, the connected subspace Y_α must be contained in U or in V. For each $\alpha \in A$, $x \in Y_\alpha \cap U$ implies that $Y_\alpha \not\subseteq V$, so we must have $Y \subseteq U$. □

Proposition 1.3.13. *Let X be a topological space.*

(1) *The connected components of X are disjoint.*

(2) *Each nonempty connected subspace $Y \subseteq X$ is contained in a unique connected component. In particular, X is the union of its connected components.*

Proof. Consider Claim (1). Suppose that two connected components X_0 and X_1 of X intersect in a point x. By Proposition 1.3.12, the union $X_0 \cup X_1$ is connected. By maximality of connected components, we must have $X_0 = X_0 \cup X_1 = X_1$.

Consider Claim (2). Let $Z \subseteq X$ be a nonempty connected subspace. Let A be a totally ordered set, let Y_α be a connected subspace of X containing Z for each $\alpha \in A$, and suppose that $Y_\alpha \subseteq Y_\beta$ for each $\alpha \leq \beta$ in A. The union $Y \coloneqq \bigcup_{\alpha \in A}$ is connected and contains Z. Indeed, if U and V are disjoint open subsets of Y whose union is Y, then $Y_\alpha \subseteq U$ or $Y_\alpha \subseteq V$ for each $\alpha \in A$. Let $\alpha \in A$ and suppose without loss of generality that $Y_\alpha \in U$. As U and V are disjoint, as U and V are disjoint. It follows that $Y \subseteq U$. This shows that the union Y is connected. Zorn's lemma now implies that the set of connected subspaces of X containing Z admits maximal elements with respect to inclusion. Such maximal elements are connected components, so Z is contained in a connected component. For the second assertion, note that each $x \in X$ is contained in the connected subspace $\{x\}$. □

Hausdorff spaces

Definition 1.3.14. The topological space X is *Hausdorff* if, for each $x, y \in X$ such that $x \neq y$, there exist open neighborhoods $x \in U \subseteq X$ and $y \in V \subseteq X$ such that $U \cap V = \varnothing$.

Example 1.3.15. $\mathbf{R}^n$, $\mathbf{R}^1$ with doubled origin

Remark 1.3.16. As illustrated by the real line with doubled origin, limits are not well defined in non-Hausdorff spaces, so there is little hope of transporting the fundamental concepts of calculus to such spaces.

Proposition 1.3.17. *Let X be a topological space and let*

$$\Delta \coloneqq \{(x, x) \in X \times X \mid x \in X\} \subseteq X \times X$$

denote the diagonal. *The subset $\Delta \subseteq X \times X$ is closed with respect to the product topology if and only if X is Hausdorff.*

Proof. The subset $\Delta \subseteq X \times X$ is closed if and only if the complement

$$X \times X - \Delta = \{(x, y) \in X \times X \mid x \neq y\}$$

is open. By the proof of Proposition 1.2.1, $X \times X - \Delta$ is open if and only if it is a union of sets of the form of the form $U \times V$ with $U, V \subseteq X$ open. It remains to observe that, if $W, W' \subseteq X$ are subsets, then $W \cap W' = \varnothing$ if and only if $W \times W' \subseteq X \times X - \Delta$. Indeed, $x \in W \cap W'$ if and only if $(x, x) \in W \times W'$. $\square$

Proposition 1.3.18. *If X is a Hausdorff space and $x \in X$, then $\{x\} \subseteq X$ is closed.*

Proof. If X is Hausdorff, then each $y \in X - \{x\}$ belongs to an open subset U_y such that $x \notin U_y$. It follows that $X - \{x\} = \bigcup_{y \in X - \{x\}} U_y$ is open. $\square$

Proposition 1.3.19. *If X is a Hausdorff space and $Y \subseteq X$ is a subspace, then Y is Hausdorff.*

Proof. Let $x, y \in Y$. Choose disjoint open neighborhoods $x \in U \subseteq X$ and $y \in V \subseteq X$. The subsets $U \cap Y$ and $V \cap Y$ are open in the subspace topology, disjoint, and contain x and y, respectively. $\square$

Proposition 1.3.20. *Let X and Y be Hausdorff spaces. The product $X \times Y$ is Hausdorff.*

Proof. Let (x, y) and (x', y') be distinct points of $X \times Y$. If $x = x'$, then choose disjoint open neighborhoods $y \in V$ and $y' \in V'$ in Y. The subsets $X \times V$ and $X \times V'$ are disjoint open neighborhoods in $X \times Y$. If $x \neq x'$, then choose disjoint open neighborhoods $x \in U$ and $x' \in U'$ in X. The subsets $U \times Y$ and $U' \times Y'$ are disjoint open neighborhoods in $X \times Y$. $\square$

Quasi-compact spaces

Definition 1.3.21. Let X be a topological space. An *open cover* $\{U_i\}_{i \in I}$ *of* X consists of a family of open subsets $U_i \subseteq X$ indexed by a set I such that $X = \bigcup_{i \in I} U_i$.

Example 1.3.22. Let $n \in \mathbf{Z}_{\geq 0}$ and $\varepsilon \in \mathbf{R}_{>0}$. The set of open disks of the form $\mathbf{D}(x, \varepsilon) \subseteq \mathbf{R}^n$ with $x \in \mathbf{Q}^n$ is an open cover. Indeed, each element of this set is open by definition, and $\mathbf{Q}^n$ is dense in $\mathbf{R}^n$, so each $y \in \mathbf{R}^n$ belongs to at least one of these open sets.

Definition 1.3.23. A topological space X is *quasi-compact* if, for each open cover $\{U_i\}_{i \in I}$ of X, there exists a finite subset $J \subseteq I$ such that $\{U_j\}_{j \in J}$ is an open cover of X.

Example 1.3.24. Consider the open cover $\{U_k\}_{k\in\mathbf{Z}}$ of $\mathbf{R}$ such that U_k is the interval $(k-1, k+1)$ of radius 1 centered at the integer k. A finite subset of $\{U_k\}_{k\in\mathbf{Z}}$ consisting of N elements of this set will cover a length of magnitude less than $2N$, and is therefore not a cover. This shows that $\mathbf{R}$ is not quasi-compact with respect to the Euclidean topology.

Lemma 1.3.25. *Let X be a topological space and let $K, K' \subseteq X$ be quasi-compact subspaces. The union $K \cup K'$ is quasi-compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of $K \cup K'$. As K and L are quasi-compact and contained in $\bigcup_{i\in I} U_i$, we can find finite subcovers $\{U_j\}_{j\in J}$ and $\{U_{j'}\}_{j'\in J'}$ of K and K', respectively. The union $\{U_j\}_{j\in J} \cup \{U_{j'}\}_{j'\in J'}$ is a finite subcover of $K \cup K'$. □

Proposition 1.3.26. *For each $a, b \in \mathbf{R}$, the subspace $[a, b] \subseteq \mathbf{R}$ is quasi-compact.*

Proof. If $b < a$, then $[a, b] = \varnothing$ is quasi-compact: the empty subcover $\varnothing$ is a finite subcover. If $a = b$, then $[a, b] = \{a\}$ is quasi-compact: $a \in U_i$ for some $i \in I$, in which case $\{U_i\}$ is a finite subcover. We may therefore assume without loss of generality that $a < b$.

Let $\{U_i\}_{i\in I}$ be an open cover of $J_0 := [a, b]$. Suppose for contradiction that $\{U_i\}_{i\in I}$ does not admit a finite subcover. If $a' = (a + b)/2$ is the midpoint, then $\{U_i\}_{i\in I}$ does not admit a finite subcover over at least one of the subintervals $[a, a']$ and $[a', b]$. Denote this non-quasi-compact sub-interval by J_1. Repeat this process, inductively subdividing J_k into two halves for each $k \in \mathbf{Z}_{\geq 0}$, each of which will be of length $(b - a)/2^k$, and choose a one J_{k+1} among them over which our cover admits no finite subcover. For each $k \in \mathbf{Z}_{\geq 0}$, let $x_k \in J_k$. The sequence $(x_k)_{k\geq 0}$ is a Cauchy with a limit $x = \lim_{k\to\infty} x_k$. This limit must belong to $J_\infty := \bigcap_{k\geq 0} J_k$. Choose $i_0 \in I$ such that $x \in U_{i_0}$. As U_{i_0} is open, there exists $\varepsilon \in \mathbf{R}_{>0}$ such that $\mathbf{D}(x, \varepsilon) \subseteq U_{i_0}$. For sufficiently large k, $J_k \subseteq \mathbf{D}(x, \varepsilon) \subseteq U_{i_0}$. This contradicts our hypothesis that the open cover admits no finite subcover over each T_k. □

Proposition 1.3.27. *If K is a quasi-compact space and $Z \subseteq K$ is a closed subspace, then Z is quasi-compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of Z. By definition of the subspace topology, for each $i \in I$, there exists an open set $V_i \subseteq K$ such that $V_i \cap Z = U_i$. The complement $K - Z$ is open by hypothesis. The family $\{K - Z\} \cup \{V_i\}_{i\in I}$ is an open cover of K. By quasi-compactness, this open cover admits a finite subcover, which must be of the form $\{K - Z, V_{i_1}, \ldots, V_{i_n}\}$ for some $i_1, \ldots, i_n \in I$. This finite subcover contains only finitely many of the V_i. By construction, it follows that $\{U_{i_1}, \ldots, U_{i_n}\}$ is an open cover of Z. □

Proposition 1.3.28. *Let $f\colon X \to Y$ be a continuous map of topological spaces. If X is quasi-compact, then $f(X)$ is quasi-compact.*

Proof. The map f factors uniquely through a surjective, continuous map $X \to f(X)$, so we may assume without loss of generality that $f(X) = Y$. Let $\{U_i\}_{i\in I}$ be an open cover of Y. The family $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of X. As X is quasi-compact, there exists a finite subset $J \subseteq I$ such that $\{f^{-1}(U_j)\}_{j\in J}$ covers X. We claim that $\{U_j\}_{j\in J}$ is an open cover of Y. Let $y \in Y$. By surjectivity of f, there exists $x \in X$ such that $f(x) = y$. The element x belongs to $f^{-1}(U_j)$ for some $j \in J$, so $y = f(x) \in f(f^{-1}(U_j)) \subseteq U_j$. □

Corollary 1.3.29. *If X is a quasi-compact topological space and R is an equivalence relation on X, then the quotient space X/R is quasi-compact.*

Proof. By Proposition 1.2.6, the quotient map $p\colon X \to X/R$ is surjective, so this is a special case of Proposition 1.3.28. □

Proposition 1.3.30. *Let X and Y be quasi-compact topological spaces. The product $X \times Y$ is quasi-compact.*

Proposition 1.3.31. *If X is a Hausdorff space and $K \subseteq X$ is a quasi-compact subspace, then K is closed in X.*

Proof. Suppose for contradiction that K is not closed in X. By Proposition 1.1.25, there exists a limit point x of K contained in $X - K$. For each $y \in K$, use the Hausdorff hypothesis to choose disjoint open neighborhoods $x \in U_y$ and $y \in V_y$. We have an open cover $K \subseteq \bigcup_{y \in K} V_y$. By quasi-compactness, we may choose a finite subcover $\{V_{y_1}, \ldots, V_{y_n}\}$. Observe that $U_{y_1} \cap \cdots \cap U_{y_n}$ is an open neighborhood of x in K disjoint from $K \cap (U_{y_1} \cup \cdots \cup U_{y_n}) = K$, which contradicts the hypothesis that x is a limit point of K. □

Definition 1.3.32. Let $n \in \mathbf{Z}_{\geq 0}$ and let $X \subseteq \mathbf{R}^n$. We say that X is *bounded* if there exists $r \in \mathbf{R}_{>0}$ such that $|x - y| \leq r$ for each $x, y \in X$.

Theorem 1.3.33 (Heine-Borel)**.** *Let $n \in \mathbf{Z}_{\geq 0}$. A subspace $K \subseteq \mathbf{R}^n$ is quasi-compact if and only if K is closed and bounded in $\mathbf{R}^n$.*

Proof. Suppose that K is quasi-compact. By Proposition 1.3.31, K is closed in $\mathbf{R}^n$. We claim that K is also bounded. Let $x \in K$. For each $r \in \mathbf{Z}_{>0}$, let $U_r := \mathbf{D}(x, r)$ denote the open disk of radius r centered at x. The family $\{U_r\}_{r \geq 0}$ is an open cover of K. By quasi-compactness, there is a finite subcover $\{U_{r_1}, \ldots, U_{r_n}\}$. If r denotes the supremum of $r_1, \ldots, r_n$, then $K \subseteq U_{r_1} \cup \cdots \cup U_{r_n} \subseteq U_r$. Thus, $|x - y| < 2r$ for each $x, y \in K$.

Conversely, suppose that K is closed and bounded. Boundedness implies that K is contained in a product $[a, b]^n$ for suitable values of $a, b \in \mathbf{R}$. By Proposition 1.3.30 and Proposition 1.3.26, $[a, b]^n$ is quasi-compact. By Proposition 1.3.27, it follows that K is quasi-compact. □

Example 1.3.34. Let $n \in \mathbf{Z}_{\geq 0}$. The function $x \mapsto |x| \colon \mathbf{R}^{n+1} \to \mathbf{R}$ is continuous, so the unit n-sphere $\mathbf{S}^n \subseteq \mathbf{R}^{n+1}$ is closed: it is the preimage of the closed subset $\{1\} \subseteq \mathbf{R}$ under a continuous map. It is also bounded by definition. By Theorem 1.3.33, $\mathbf{S}^n$ is therefore quasi-compact.

Alternatively, the space $[-1, 1]$ is quasi-compact by Proposition 1.3.26; the finite product $[-1, 1]^{n+1} \subseteq \mathbf{R}^{n+1}$ is therefore also quasi-compact by Proposition 1.3.30; and $\mathbf{S}^n$ is quasi-compact as a closed subspace of the quasi-compact space $[-1, 1]^{n+1}$ by Proposition 1.3.27.

Example 1.3.35. The space $\mathbf{CP}^n$ is quasi-compact for each $n \in \mathbf{Z}_{\geq 0}$. The subspace $\mathbf{S}^{2n+1} \subseteq \mathbf{C}^{n+1} - \{0\}$ is quasi-compact by Example 1.3.34, and its image under the quotient map $\mathbf{C}^{n+1} - \{0\} \to \mathbf{CP}^n$ is $\mathbf{CP}^n$, so the claim follows from Proposition 1.3.28.

Theorem 1.3.36 (Cantor)**.** *Let X be a topological space and let $\cdots \subseteq K_2 \subseteq K_1 \subseteq K_0 \subseteq X$ be a nested sequence of nonempty, closed, quasi-compact subsets. The intersection $K_\infty := \bigcap_{n \in \mathbf{Z}_{\geq 0}} K_n$ is nonempty.*

Proof. For each $n \in \mathbf{Z}_{>0}$, let $U_n := K_0 - K_n$. Intersections of closed subsets are closed, so U_n is open in K_0 for each n. As the K_n are nested, so are their complements: $U_1 \subseteq U_2 \subseteq \cdots$. Suppose that $K_\infty = \varnothing$ or, equivalently, that $\{U_n\}_{n>0}$ is an open cover of K_0. It suffices to deduce that $K_N = \varnothing$ for some $N \in \mathbf{Z}_{>0}$ or, equivalently, that $U_N = K_0$. As K_0 is quasi-compact, there is a finite subcover. As the U_n are nested, there exists $N \in \mathbf{Z}_{>0}$ such that $U_N = K_0$: take N to be the supremum of the indices of the elements of the finite subcover. □

1.4 Topological groups

Definition 1.4.1. A *topological monoid* (M, μ) consists of a topological space M equipped with a continuous map $\mu \colon M \times M \to M$ with respect to the product topology on the domain, such that M is a monoid with respect to the binary operation $(x, y) \mapsto x \cdot y := \mu(x, y)$, i.e., μ is associative and unital.

Example 1.4.2. Consider the set $M \coloneqq \mathrm{M}_{n\times n}(\mathbf{C})$ of $n \times n$ matrices with complex entries. With respect to matrix multiplication, M is a monoid. Identify M with $\mathbf{C}^{n^2}$ and equip it with the Euclidean topology. To see that matrix multiplication is continuous with respect to this topology, observe that the entries of the product of two matrices are polynomial expressions in the entries of the matrices being multiplied, and polynomial maps are continuous.

Definition 1.4.3. A *topological group* (G, μ, ι) consists of a topological monoid (G, μ) whose underlying monoid is a group, equipped with a continuous map $\iota\colon G \to G$ such that $\iota(g) = g^{-1}$ for each $g \in G$.

Example 1.4.4. Consider the set $G \coloneqq \mathrm{GL}_n(\mathbf{C})$ of invertible $n \times n$ matrices with complex entries. With respect to matrix multiplication, G is a group. With the notation of Example 1.4.2, equip $G \subseteq M$ with the subspace topology. The inclusion is a morphism of monoids, so the multiplication map $\mu\colon G \times G \to G$ is continuous. The inversion map $\iota\colon g \mapsto g^{-1}\colon G \to G$ is given by polynomial expressions in the entries of the input matrix divided by the determinant, which is a nonvanishing polynomial expression in the entries of the input matrix. The inversion map is therefore continuous.

Definition 1.4.5. Let G be a topological group and X a topological space. A *continuous action of G on X* is a continuous map $\rho\colon G \times X \to X$ with respect to the product topology such that ρ is an action of the group underlying G on the set underlying X, i.e., such that ρ is associative and unital.

Example 1.4.6. Each topological monoid M acts continuously on itself via the multiplication map $\mu\colon M \times M \to M$.

Example 1.4.7. The topological monoid $\mathrm{M}_{n\times n}(\mathbf{C})$ acts continuously on $\mathbf{C}^n$ if we identify elements of $\mathbf{C}^n$ with column vectors and let elements of $\mathrm{M}_{n\times n}(\mathbf{C})$ act by matrix multiplication. Indeed, the action map $\mathrm{M}_{n\times n}(\mathbf{C}) \times \mathbf{C}^n \to \mathbf{C}^n$ is again given by polynomial expressions in the entries of the inputs, and is therefore continuous.

Example 1.4.8. By restriction of the action map of Example 1.4.7, we obtain a continuous action of $\mathrm{GL}_n(\mathbf{C})$ on $\mathbf{C}^n$.

Moreover, left multiplication by elements of $\mathrm{GL}_n(\mathbf{C})$ sends the subspace $\mathbf{C}^n - \{0\}$ to itself, as invertible matrices have trivial kernels. We therefore also have a continuous group action of $\mathrm{GL}_n(\mathbf{C})$ on $\mathbf{C}^n - \{0\}$ for each $n > 0$.

We may furthermore identify $\mathbf{C}^* = \mathrm{GL}_1(\mathbf{C})$ with the subspace of $\mathrm{GL}_n(\mathbf{C})$ spanned by the nonzero multiples of the identity matrix. This subspace is a topological subgroup, and we deduce a continuous group action of $\mathbf{C}^*$ on $\mathbf{C}^n - \{0\}$ for each $n > 0$.

Definition 1.4.9. Let G be a topological group, let X be a topological space, and let $\rho\colon G\times X \to X$ be a continuous group action. The orbits $Gx \coloneqq \{\rho(g, x) \in X \mid g \in G\} \subseteq X$ for $x \in X$ are the equivalence classes for an equivalence relation on X. The *quotient of X by G* is the quotient of X by this equivalence relation, equipped with the quotient topology. In particular, it admits a continuous surjection $\pi\colon X \to X/G$.

Example 1.4.10. Let $n \in \mathbf{Z}_{\geq 0}$. By Example 1.4.8, the topological group $\mathbf{C}^*$ acts on $\mathbf{C}^{n+1} - \{0\}$. Two points $z, w \in \mathbf{C}^{n+1} - \{0\}$ belong to the same orbit for this action if and only if they are equivalent with respect to the relation used to define $\mathbf{CP}^n$ (Example 1.2.9). It follows that $\mathbf{CP}^n$ is equal to the quotient of $\mathbf{C}^{n+1} - \{0\}$ by the action of $\mathbf{C}^*$.

Lemma 1.4.11. *Let G be a topological group, X a topological space, and $\rho\colon G \times X \to X$ a continuous group action. For each $g \in G$ and each open subset $U \subseteq X$, the subset $gU \coloneqq \{\lambda(g, x) \mid x \in X\} \subseteq X$ is open.*

Proof. For each $g \in G$, the composite

$$\rho_g \colon X \simeq \{g\} \times X \hookrightarrow G \times X \xrightarrow{\rho} X$$

is continuous. The definition of a topological group implies that this composite is a homeomorphism with continuous inverse $\rho_{g^{-1}}$. The claim follows, as gU is the image of U under the homeomorphism ρ_g. □

Proposition 1.4.12. *Let G be a topological group, X a topological space, and $\rho\colon G \times X \to X$ a continuous group action. The quotient map $\pi\colon X \to X/G$ is an open map.*

Proof. Let $U \subseteq X$ be open. We claim that $\pi(U)$ is open in X/G. By definition of the quotient topology, $\pi(U)$ is open if and only if $\pi^{-1}(\pi(U))$ is open. We have

$$\pi^{-1}(\pi(U)) = \{x \in X \mid \pi(x) \in \pi(U)\} = \{x \in X \mid \exists y \in U\,[(x, y) \in R]\} = \bigcup_{g \in G} gU,$$

where R is the equivalence relation of Definition 1.4.9. By Lemma 1.4.11, the last expression is a union of open subsets gU, hence open. □

Proposition 1.4.13. *The space* $\mathbf{CP}^n$ *is Hausdorff for each* $n \in \mathbf{Z}_{\geq 0}$.

Proof. By Proposition 1.3.17, it suffices to show that the diagonal $\Delta \subseteq \mathbf{CP}^n \times \mathbf{CP}^n$ is closed with respect to the product topology or, equivalently, that its complement is open. Consider the map

$$(z, w) \mapsto \sum_{0 \leq k \neq \ell \leq n+1} |z_k w_\ell - z_\ell w_k|^2 \colon (\mathbf{C}^{n+1} - \{0\}) \times (\mathbf{C}^{n+1} - \{0\}) \to \mathbf{R}.$$

This is a continuous map.

For each $z, z', w, w' \in \mathbf{C}^{n+1} - \{0\}$, if $z \sim z'$ and $w \sim w'$, then $f(z, w) = f(z', w')$, where $\sim$ is the equivalence relation used to define $\mathbf{CP}^n$ (Example 1.2.9). It follows that f factors uniquely as a composite of the form

$$(\mathbf{C}^{n+1} - \{0\}) \times (\mathbf{C}^{n+1} - \{0\}) \xrightarrow{\pi \times \pi} \mathbf{CP}^n \times \mathbf{CP}^n \xrightarrow{\bar{f}} \mathbf{R},$$

where $\pi\colon \mathbf{C}^{n+1} - \{0\} \to \mathbf{CP}^n$ is the quotient map and $\bar{f}$ is continuous.

For each $z, w \in \mathbf{C}^{n+1} - \{0\}$, if $f(z, w) = 0$, then, for each $0 \leq k \neq \ell \leq n + 1$, we have $z_k w_\ell = z_\ell w_k$. It follows that z and w are collinear, i.e., z and w belong to the same orbit for the action of $\mathbf{C}^*$ on $\mathbf{C}^{n+1} - \{0\}$. By Example 1.4.10, $\mathbf{CP}^n$ is the quotient of $\mathbf{C}^{n+1} - \{0\}$ by this action. Combining these observations, the diagonal of $\mathbf{CP}^n$ is the image of $f^{-1}(\{0\})$ under $\pi \times \pi$. As $\pi \times \pi$ is surjective, this is equivalent to the condition that the image of $f^{-1}(\mathbf{R} - \{0\})$ under $\pi \times \pi$ be equal to the complement of the diagonal. As $\mathbf{R} - \{0\} \subseteq \mathbf{R}$ is open, and f is continuous, it suffices to show that $\pi \times \pi$ is open. By Proposition 1.4.12, the quotient map π is open. It therefore follows from the proof of Proposition 1.2.1 that $\pi \times \pi$ is also open. □

1.5 Proper maps

Definition 1.5.1. Let $f\colon X \to Y$ be a function between topological spaces. We say that f is *proper* if, for each quasi-compact $K \subseteq Y$, the preimage $f^{-1}(K) \subseteq X$ is also quasi-compact.

Example 1.5.2. A topological space X is quasi-compact if and only if the unique map $X \to *$ is proper.

Example 1.5.3. The quotient map $\mathbf{R} \to \mathbf{R}/\mathbf{Z} \simeq \mathbf{S}^1$ is not proper. Indeed, $\mathbf{S}^1$ is quasi-compact by [add reference], while its preimage $\mathbf{R}$ is not by [add reference].

Example 1.5.4. If X is a topological space and $j\colon U \hookrightarrow X$ is the inclusion of an open subspace, then j will in general not be proper. Indeed, $j^{-1}(K) = U \cap K$ need not be quasi-compact for a general quasi-compact subspace $K \subseteq X$.

Example 1.5.5. For each topological space X, id_X is proper.

Example 1.5.6. If $f\colon X \to Y$ and $g\colon Y \to Z$ are proper maps, then $g \circ f\colon X \to Z$ is proper.

Exercise 1.5.7. Let $f\colon X \to Y$ be a proper, continuous map and let $V \subseteq Y$ be an open subspace. The map $f\colon f^{-1}(V) \to V$ is proper.

Definition 1.5.8. Let X be a topological space. We say that X is *compactly generated* if, for each subset $S \subseteq X$, S is closed if and only if, for each quasi-compact $K \subseteq X$, $S \cap K$ is closed in K.

Proposition 1.5.9. *Each locally Euclidean topological space X is compactly generated.*

Proof. Let $S \subseteq X$ be a subset such that $S \cap K$ is closed for each quasi-compact subspace $K \subseteq X$. We claim that each $x \in X - S$ admits an open neighborhood contained in $X - S$. Choose an open neighborhood U of x equipped with a homeomorphism $\varphi\colon U \to V$ with V an open subset of $\mathbf{R}^n$ for some $n \in \mathbf{Z}_{\geq 0}$. There exists a closed disk $D \coloneqq \bar{\mathbf{D}}(\varphi(x), r)$ of radius $r \in \mathbf{R}_{>0}$ such that $\bar{\mathbf{D}}(\varphi(x), r) \subseteq V$. By [add reference], D is quasi-compact. It follows that $K \coloneqq \varphi^{-1}(D) \subseteq U$ is a quasi-compact neighborhood of x. Let $x \in V \subseteq X$ be open such that $V \subseteq K$. By hypothesis, $K - S$ is open in K, and $U \coloneqq V \cap (K - S)$ is therefore open in $V \cap K = V$ and, hence, in X. This subset U is the required open neighborhood of x. □

Proposition 1.5.10. *Let $f\colon X \to Y$ be a continuous map of topological spaces.*

(1) *If f is closed with quasi-compact fibers, then f is proper.*
(2) *If f is proper and Y is compactly generated and Hausdorff, then f is closed.*
(3) *If X is quasi-compact and Y is Hausdorff, then f is closed and proper.*

Proof. Consider Claim (1). Let $K \subseteq Y$ be quasi-compact and let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of $f^{-1}(K)$. We seek a finite subcover. For each $k \in K$, $\mathfrak{U}$ is an open cover of $f^{-1}(k)$. By hypothesis, $f^{-1}(k)$ is quasi-compact, so we may choose a finite subcover corresponding to a finite subset $A_k \subseteq A$. As f is closed, $f(X - \bigcup_{\alpha \in A_k} U_\alpha)$ is closed in Y or, equivalently, $V_k \coloneqq Y - f(X - \bigcup_{\alpha \in A_k} U_\alpha)$ is open in Y. By construction, $k \in V_k$ for each $k \in K$, so $\{V_k\}_{k \in K}$ is an open cover of K. As K is quasi-compact, there exists a finite subcover $\{V_{k_r}\}_{1 \leq r \leq n}$. Thus, $\{f^{-1}(V_{k_r})\}_{1 \leq r \leq n}$ is an open cover of $f^{-1}(K)$. For each $1 \leq r \leq n$, $f^{-1}(V_k) \subseteq \bigcup_{\alpha \in A_{k_r}} U_\alpha$, so $\{U_\alpha\}_{\alpha \in \bigcup_{1 \leq r \leq n} A_{k_r}}$ is the desired finite subcover.

Consider Claim (2). Let $Z \subseteq X$ be closed and let $K \subseteq Y$ be quasi-compact. As Y is compactly generated, it suffices to show that $K \cap f(Z)$ is closed in K. By hypothesis, $f^{-1}(K)$ is quasi-compact. By [add reference], it follows that $f^{-1}(K) \cap Z$ is quasi-compact. By [add reference], $f(f^{-1}(K) \cap Z) = K \cap f(Z)$ is quasi-compact. As Y is Hausdorff, $K \cap f(Z)$ is closed in K by [add reference].

Consider Claim (3). Let $Z \subseteq X$ be closed. By [add reference], it follows that Z is quasi-compact. By [add reference], $f(Z)$ is therefore quasi-compact. As Y is Hausdorff, this implies that $f(Z)$ is closed by [add reference].

Let $K \subseteq Y$ be quasi-compact. As Y is Hausdorff, this implies that K is closed by [add reference]. By continuity of f, $f^{-1}(K)$ is therefore closed. By [add reference], $f^{-1}(K)$ is quasi-compact. □

References