Topology of algebraic varieties
Exercise sheet 1

Definition. Let X be a topological space, $Y \subseteq X$ a subset, and $x \in X$. We say that x is an *accumulation point of Y in X* if, for each neighborhood V of x, we have $(Y \cap V) - \{x\} \neq \varnothing$.

Exercise 1.1. Give an example of a holomorphic function $f\colon \mathbf{C}^2 \to \mathbf{C}$ and a subset $X \subseteq \mathbf{C}^2$ with an accumulation point x such that $f|_X = 0$ while f is not identically zero, or prove that no such function exists.

Exercise 1.2. Give an example of a nonconstant, holomorphic map $f\colon \mathbf{C}^2 \to \mathbf{C}^2$ that is not an open mapping, or prove that no such map exists.

Definition. Let $X \subseteq \mathbf{C}^n$. An *analytic disk in X* is a nonconstant analytic map $f\colon \mathbf{D}(0,1) \to X$, where $\mathbf{D}(0,1)$ is the open disk of radius 1 centered at the origin in $\mathbf{C}$.

Exercise 1.3. The unit sphere $\mathbf{S}^{2n-1} \subseteq \mathbf{C}^n$ contains no analytic disks.

Definition. Let $r \in \mathbf{Z}_{>0}$. A function $f\colon \mathbf{C}^n \to \mathbf{C}$ is *homogeneous of degree r* if

$$f(\lambda z) = \lambda^r f(z)$$

for each $\lambda \in \mathbf{C}$ and each $z \in \mathbf{C}^n$.

Exercise 1.4. Let $r \in \mathbf{Z}_{>0}$, let $f\colon \mathbf{C}^n \to \mathbf{C}$ be a holomorphic function, and prove Euler's identity

$$rf(a) = \sum_{k=1}^{n} a_k \frac{\partial f}{\partial z_k}(a)$$

for each $a \in \mathbf{C}^n$.

Exercise 1.5 (Riemann extension theorem)**.** Let $U \subseteq \mathbf{C}^n$ be a connected, open subset, let $g\colon U \to \mathbf{C}$ be a nonconstant holomorphic function, let $Z := g^{-1}(0) \subseteq \mathbf{C}^n$ be the vanishing locus of g, and let $f\colon U - Z \to \mathbf{C}$ be a holomorphic function. Show that there exists a unique holomorphic function $F\colon U \to \mathbf{C}$ such that $F|_{U-Z} = f$.

Exercise 1.6. Let X be a Hausdorff topological space, let $\sim$ be an equivalence relation on X, and suppose that the quotient map $\pi\colon X \to X/\sim$ is an open mapping with respect to the quotient topology. Show that $X/\sim$ is Hausdorff if and only if the set

$$\{(x,y) \in X \times X \mid x \sim y\}$$

is closed in $X \times X$.

Exercise 1.7. Let G be a topological group, X a topological space, and $\mu\colon G \times X \to X$ a continuous action. Define an equivalence relation on X by $x \sim y$ if and only if there exists $g \in G$ such that $g \cdot x = y$ and let $\pi\colon X \to X/G$ denote the associated quotient map. Show that π is an open map with respect to the quotient topology.

Exercise 1.8. Show that $\mathbf{CP}^n$ is Hausdorff by considering the map

$$f\colon (x,y) \mapsto \sum_{k \neq \ell} |x_k y_\ell - x_\ell y_k|^2 \colon (\mathbf{C}^{n+1} - \{0\}) \times (\mathbf{C}^{n+1} - \{0\}) \to \mathbf{R}.$$